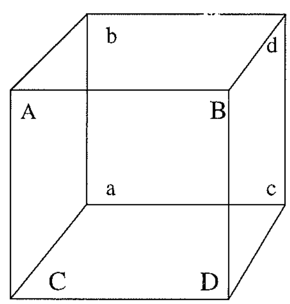
Algebraic Topology Qualifying Exam, Winter 2005

(1) Classify, up to homeomorphism, all spaces which cover the torus $T = S^1 \times S^1$. You do *not* need to classify the covering maps.

(2) Use covering space theory to show that any subgroup of a free group is free, and explicitly identify a subgroup of the free group on two generators which is isomorphic to the free group on five generators. Bonus points for having a normal subgroup.

(3) Let X be the space one obtains when one takes a (solid) cube and identifies pairs of opposite faces as follows. The front and back faces are identified as in the figure below, so that corner A is identified with corner a and so forth. In other words, points are identified if they are related by a projection followed by a ninety degree clockwise rotation, from the point of view of the front face.



Identify the right and left, and top and bottom faces similarly, through projections followed by ninety degree clockwise rotations, from the perspective of the right and top faces.
(a) Show that X is a CW-complex; you may assume that the cube is a CW-complex in the standard way. Is X a manifold (only a yes or no is required)?
(b) Explicitly describe the cellular chain complex of X.
Bonus: Compute the homology of X.

(4) Consider the assignment sending a space X to the graded abelian group $k_*(X) = \tilde{H}_*(X) \otimes \mathbb{Z}/53$. Determine whether this assignment satisfies each axiom for a (reduced) homology theory.

(5) Let C_*, for $1 \leq * \leq N$, be a chain complex of vector spaces over a field which is finite dimensional. Show that $\Sigma_{i=1}^N (-1)^i \text{rank } C_i = \Sigma_{i=1}^N (-1)^i \text{rank } H_i(C_*)$.

(6) (a) Compute the homology of X, the space obtained by identifying the boundary S^1 of a Möbius band with the diagonal $\Delta = \{(x, x)\} \subset S^1 \times S^1$.
(b) Compute the homology of Y, obtained by identifying the boundary S^1 of a Möbius band with $S^1 = \mathbb{R}P^1 \subset \mathbb{R}P^2$.

(7) Let Q be the quotient space of $\mathbb{C}P^2 \times \mathbb{C}P^2$ obtained by setting $* \times x \sim x \times *$ where $*$ is a fixed point (say $[1,0,0]$ in homogeneous coordinates, to be definite) and x is an arbitrary point in $\mathbb{C}P^2$.
(a) Compute the homomorphism $q_* : H_*(\mathbb{C}P^2 \times \mathbb{C}P^2) \to H_*(Q)$. You may give, without proof, CW structures on $\mathbb{C}P^2 \times \mathbb{C}P^2$ and Q.
(b) Find all relations among products of additive generators in the cohomology ring of Q. [Hint: you may assume q^* is injective.]

(8) (a) State and prove the naturality property for cap products.
(b) Let $H_c^*(X) = \lim_K H^*(X, X-K)$, where K ranges over compact sets and the maps of the directed system are the standard ones induced by inclusions at the cohain level. Let M be an oriented m-dimensional manifold. Carefully define the duality map $D_M : H_c^k(M) \to H_{m-k}(M)$ and state the Poincaré duality theorem.
(c) Deduce the standard version of the Poincaré duality theorem for oriented compact manifolds from that stated in part (b).

(9) Show that $\mathbb{C}P^2$ is not the boundary of a five-dimensional manifold.

Solutions

Algebraic Topology Qualifying Exam, Winter 2005

(1) Classify, up to homeomorphism, all spaces which cover the torus $T = S^1 \times S^1$. You do *not* need to classify the covering maps.

Solution: The classification of covering spaces identifies the set of covering maps over X with conjugacy classes of subgroups of $\pi_1(X, *)$ for connected, semi-locally simply connected spaces X. We use two fundamental facts about π_1, namely that $\pi_1(S^1, 0) \cong \mathbb{Z}$ and $\pi_1(X \times Y, *_X \times *_Y) \cong \pi_1(X, *_X) \times \pi_1(Y, *_Y)$, to compute that $\pi_1(T, 0) \cong \mathbb{Z}^2$. Moreover, since the universal over of S^1 is $\mathbb{R}$, the universal cover of T is $\mathbb{R}^2$, where the covering map is reduction by the subgroup $\mathbb{Z} \times \mathbb{Z}$, which is acting as the group of deck transformations. Any cover of T will be, up to covering space isomorphism, the quotient of $\mathbb{R}^2$ by a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

A subgroup S of $\mathbb{Z}^2$ will be free abelian, of rank zero, one, or two. If S is of rank zero, that is trivial, the covering space is of course $\mathbb{R}^2$. If S is rank one generated by some v, the quotient of $\mathbb{R}^2$ by S is homeomorphic to $S^1 \times \mathbb{R}$ (one may see this by choosing a w linearly independent from v and sending $x = av + bw$ to $(a, b) \in (\mathbb{R}/Z) \times \mathbb{R}$ and checking that this is a homeomorphism). If S is rank two, the quotient by S is homeomorphic to T.

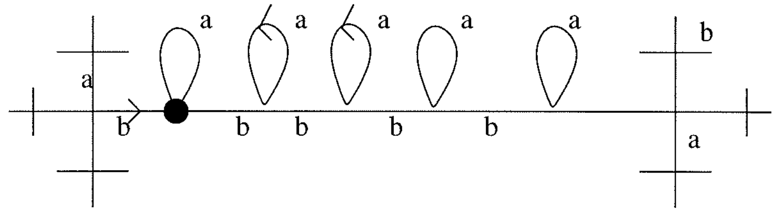
Thus, the three spaces which may cover T are $\mathbb{R}^2$, $S^1 \times \mathbb{R}$ and T.

(2) Use covering space theory to show that any subgroup of a free group is free, and explicitly identify a subgroup of the free group on two generators which is isomorphic to the free group on five generators. Bonus points for having a normal subgroup.

Solution: The fundamental theorem of covering space theory states that, for semi-locally simply connected spaces X, there is a natural bijection between based covering spaces of X and subgroups of $\pi_1(X, *)$. This bijection sends a based covering space to the (isomorphic) image of its fundamental group under the covering map.

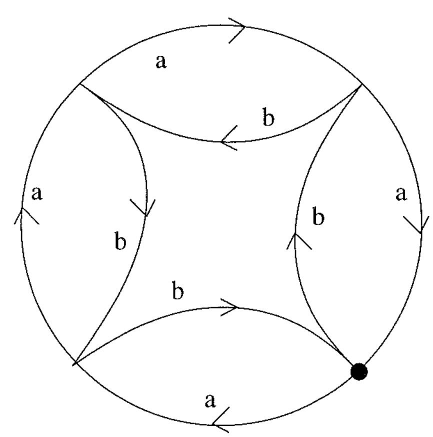
Let F_n be the free group on n generators, and realize F_n as $\pi_1(\bigvee_n S^1, *)$, where $*$ is the wedge point. We claim that any cover of F_n is a one-dimensional CW-complex. Indeed, the zero-skeleton of F_n lifts to a discrete set of points. Each one-cell lifts, by the homotopy lifting property, to union of one-cells with boundaries in the zero-skeleton. A one-dimensional CW-complex is a graph, and any graph contains a maximal tree. By quotienting this maximal tree, by identifying it all to a point, we get a wedge of circles. This wedge of circles is homotopy equivalent to the original complex because the tree is contractible and is a sub-CW-complex. The fundamental group of a wedge of circles is the free group with one generator for each wedge factor: assume this is true for k wedge factors (it is trivially true when $k = 0$) and by the Van Kampen theorem applied by noticing $\bigvee_{k+1} S^1 = \bigvee_k S^1 \cup_* S^1$, we have $\pi_1(\bigvee_{k+1} S^1 *) \cong F_k * F_1$ which is isomorphic to F_{k+1}. Thus, π_1 of any covering space for $\bigvee_n S^1$ is a free group, so the corresponding subgroup of F_n will be free.

To find subgroups of F_2 consider covers of $S^1 \vee S^1$, which correspond to four-valent graphs with edges labeled by two letters. In this language, consider the following cover of $S^1 \vee S^1$.



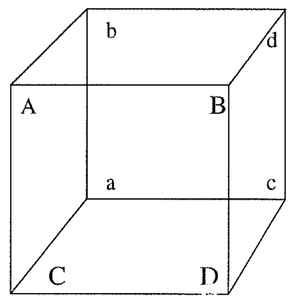
Here the graph continues to trifurcate at the right and left ends as in the universal cover. All horizontal edges are labeled by b, all vertical edges and loops by a, orientations always to the right, up, or counterclockwise, and the basepoint is as indicated. The fundamental group is isomorphic to F_5. Its image in the fundamental group of $S^1 \vee S^1$, which is F_2 say generated by a and b, is thus also isomorphic to F_5 and is generated by a, bab^{-1}, b^2ab^{-2}, b^3ab^{-3} and b^4ab^{-4}.

There are many possible answers to this question, by changing the chosen cover, or even the basepoint. One which produces a normal cover is the following.



The subgroup it corresponds to is that generated by ab, $b^{-1}ab^2$, $b^{-2}ab^3$, $b^{-3}a$, and b^4.

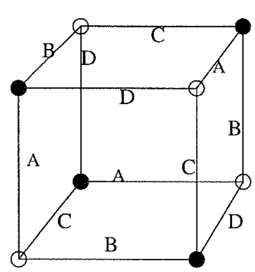
(3) Let X be the space one obtains when one takes a (solid) cube and identifies pairs of opposite faces as follows. The front and back faces are identified as in the figure below, so that corner A is identified with corner a and so forth. In other words, points are identified if they are related by a projection followed by a ninety degree clockwise rotation, from the point of view of the front face.



Identify the right and left, and top and bottom faces similarly, through projections followed by ninety degree clockwise rotations, from the perspective of the right and top faces.
(a) Show that X is a CW-complex; you may assume that the cube is a CW-complex in the standard way. Is X a manifold (only a yes or no is required)?
(b) Explicitly describe the cellular chain complex of X.
Bonus: Compute the homology of X.

Solution: (a) By inspection X is the union of subspaces each homeomorphic to open cells of various dimensions: one open 3-cell which is the image of the interior of the cube; three open 2-cells, which are the images of the interiors of the three pairs of identified faces; four 1-cells, which are the images of the interiors of edges of the cube as labeled in the ollowing diagram; two 0-cells, the images of the vertices of the cube according to whether those vertices are labeled by a solid or empty dot. Each of these open cells is the interior of a map from a closed cell to X, namely the compose of the closed cell mapping to the cube under its standard CW structure with the quotient map. Moreover, the boundary of each closed cell maps to cells of lower dimension because that was true for the cube. Thus X is a CW complex.

X is a manifold. Explanation (not required): points inside the three cell have Euclidean neighborhoods. Points on the faces have such after the identifications because two half-balls get identified. Around points on edges three "quarter-ball wedges" have boundaries identified in such a way as to have a neighborhood homeomorphic to a ball, and around the corners four "eighth-balls" are so identified.



(b) By counting cells, we have that the cellular chain groups are

$$\mathbb{Z}^2 \xleftarrow{d_0} \mathbb{Z}^4 \xleftarrow{d_1} \mathbb{Z}^3 \xleftarrow{d_2} \mathbb{Z}.$$

In the boundary of the three-cell, each two cell appears twice with opposite orientations, so $d_2 = 0$. We orient the two-cells "counterclockwise" and the one-cells so that the positive ends are always at the empty dots. Thus d_1 of the front face is $C - D + A - B$, of the right is $D - B + A - C$ and of the top is $D - A + C - B$. Finally, each one simplex has the same image under d_0, namely $E - S$, where E represents the empty zero-cell and S the solid zero-cell.

Bonus: Because $d_2 = 0$, $H_3 = \mathbb{Z}$. The cokernel of d_0 is free of rank one, generated by E or S, so $H_0 = \mathbb{Z}$. A convenient basis for the kernel of d_0, which is rank three (since the image of d_0 is rank one) is $\beta = A - B$, $\gamma = A - C$ and $\delta = A - D$. In this basis, the image of d_1 is $C - D + A - B = -\gamma + \delta + \beta$, $-\delta + \beta + \gamma$ and $-\delta - \gamma + \beta$. The corresponding matrix and its reduction by row and column operations is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus $H_2(X) = 0$, since d_1 is injective, and $H_1(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

(4) Consider the assignment sending a space X to the graded abelian group $k_*(X) = \tilde{H}_*(X) \otimes \mathbb{Z}/53$. Determine whether this assignment satisfies each axiom for a (reduced) homology theory.

Solution: k_* is a functor: For any map $f : X \to Y$, we take $f_*^k = f_*^H \otimes id : \tilde{H}_*(X) \otimes \mathbb{Z}/53 \to \tilde{H}_*(X) \otimes \mathbb{Z}/53$, where f_*^H is the induced map on homology. $(f \circ g)_*^k = f_*^k \circ g_*^k$ because

$$f_*^H \otimes id \circ g_*^H \otimes id = (f_*^H \circ g_*^H) \otimes id \circ id = (f_*^H \circ g_*^H) \otimes id.$$

Similarly id_*^k is the identity map because $id \otimes id = id$.

k_* satisfies the homotopy axiom: If $f \simeq g$ then $f_*^H = g_*^H$, so $f_*^H \otimes id = g_*^H \otimes id$ or $f_*^k = g_*^k$.

k_* satisfies the wedge axiom: if $X = \bigvee_\alpha X_\alpha$, then because homology satisfies the wedge axiom, $\oplus_\alpha \tilde{H}_*(X_\alpha) \to \tilde{H}_*(X)$ is an isomorphism. Tensoring both sides with $\mathbb{Z}/53$ preserves this isomorphism. Because the tensor product distributes over arbitrary direct sums, we have

$$\oplus_\alpha k_*(X_\alpha) = \oplus_\alpha(\tilde{H}_*(X_\alpha) \otimes \mathbb{Z}/53) \xrightarrow{\cong} \tilde{H}_*(X) \otimes \mathbb{Z}/53 = k_*(X).$$

k_* does not have a long exact sequence associated to a pair of spaces (also known as a "short exact sequence of spaces"): Let X be the Moore space $M(\mathbb{Z}/53, 1)$ obtained by attaching a 2-cell to S^1 by a degree 53 map. Let A be the image of S^1. The quotient X/A is S^2. The long exact sequence in $\tilde{H}_*$ starting at $H_2(X)$ reads

$$0 \to \mathbb{Z} \xrightarrow{\times 53} \mathbb{Z} \to \mathbb{Z}/53 \to 0.$$

When all groups involved are tensored with $\mathbb{Z}/53$ this sequence is no longer exact. In particular, the map $\times 53$ becomes the zero map on $\mathbb{Z} \otimes \mathbb{Z}/53 = \mathbb{Z}/53$, which is not injective.

(5) Let C_*, for $1 \leq * \leq N$, be a chain complex of vector spaces over a field which is finite dimensional. Show that $\Sigma_{i=1}^N(-1)^i\text{rank } C_i = \Sigma_{i=1}^N(-1)^i\text{rank } H_i(C_*)$.

Solution: We first state the following.

Lemma 1. *If* $0 \to W \to V \to V/W \to 0$ *is an exact sequence of vector spaces then* rank $V =$ rank $W +$ rank V/W.

Before proving this lemma, we use it to solve the problem. Consider the exact sequences

$$0 \to B_n \to Z_n \to H_n \to 0,$$

where B_n is the image of d_{n+1} (the boundaries) and Z_n is the kernel of d_n (the cycles), as well as

$$0 \to Z_n \to C_n \to B_{n-1}.$$

The first exact sequence expresses the definition of H_n and the second expresses the image of d_n as the quotient of its domain by its kernel. Using the second exact sequence and the lemma we have

$$\Sigma_{i=1}^N(-1)^i\text{rank } C_i = \Sigma_{i=1}^N(-1)^i(\text{rank } Z_i + \text{rank } B_{i-1}).$$

Using the second exact sequence and the lemma we can substitute for rank Z_i to obtain

$$\Sigma_{i=1}^N\text{rank } C_i = \Sigma_{i=1}^N(-1)^i(\text{rank } B_i + \text{rank } H_i + \text{rank } B_{i-1}).$$

But this sum partially telescopes, with rank B_k appearing with opposite signs when $i = k$ and $i = k + 1$. After such cancellation we have the equality to be shown.

To prove the lemma, we may show that one can take a basis of W and add pre-images of a basis for V/W to produce a basis for V.

(6) (a) Compute the homology of X, the space obtained by identifying the boundary S^1 of a Möbius band with the diagonal $\Delta = \{(x, x)\} \subset S^1 \times S^1$.
(b) Compute the homology of Y, obtained by identifying the boundary S^1 of a Möbius band with $S^1 = \mathbb{R}P^1 \subset \mathbb{R}P^2$.

Solution: (a) We use the Mayer-Vietoris sequence. Let A be the image in X of the Möbius band, B be the image in X of $S^1 \times S^1$, so that $A \cap B = \Delta \cong S^1$. All inclusions are neighborhood deformation retracts, so we may apply the Mayer-Vietoris sequence, which is zero for H_3 and above, and in lower dimensions reads

$$\cdots \to H_2(A \cap B) = 0 \to H_2(A) \oplus H_2(B) = \mathbb{Z} \to H_2(X) \to H_1(A \cap B) = \mathbb{Z} \xrightarrow{i_*} H_1(A) \oplus H_1(B) = \mathbb{Z}^3 \to H_1(X) \to \cdots.$$

The key is thus to compute i_* which is induced by the inclusions of $A \cap B$ into A and B. The inclusion into A is that of the boundary of the Möbius band, which multiplies by 2 on H_1. The inclusion into B sends a generator of H_1 to $1 \times 1 \in \mathbb{Z} \times \mathbb{Z}$. Thus i_* is injective, so that the previous map in the sequence is zero, which means the map before that was an isomorphism so that $H_2(X) \cong \mathbb{Z}$. The cokernel of i_* is $\mathbb{Z}^3$ modulo the subgroup generated by $(2,1,1)$, which is simply $\mathbb{Z}^2$. This is in fact isomorphic to $H_1(X)$ since further down the exact sequence $H_0(A \cap B)$ maps injectively under (the next) i_*.

(b) Let A be the image of the Möbius band, B of $\mathbb{R}P^2$, their intersection an S^1. Again, all subspaces are neighborhood deformation retracts. The Mayer-Vietoris sequence reads

$$\cdots \to H_2(A \cap B) = 0 \to H_2(A) \oplus H_2(B) = 0 \to H_2(Y) \to H_1(A \cap B) = \mathbb{Z} \xrightarrow{i_*} H_1(A) \oplus H_1(B) = \mathbb{Z} \oplus \mathbb{Z}/2 \to H_1(Y) \to \cdots .$$

Once again, computing i_* is the main step. It sends a generator of $H_1(S^1)$ to $(2,1) \in \mathbb{Z} \times \mathbb{Z}/2$, since as before the boundary of the Möbius band represents twice a generator of its H_1 and $\mathbb{R}P^1$ in $\mathbb{R}P^2$ represents a generator of that H_1. Since this map is injective we see that $H_2(Y) = 0$. The quotient of $\mathbb{Z} \oplus \mathbb{Z}/2$ by the image of i_* is $\mathbb{Z}/4$ (note that $(4,0)$ is in the subgroup by which we quotient and that any $(a,1)$ is congruent to $(a-2,0)$ modulo this subgroup. Thus $H_1(Y) \cong \mathbb{Z}/4$ since the map i_* is injective on H_0.

(7) Let Q be the quotient space of $\mathbb{C}P^2 \times \mathbb{C}P^2$ obtained by setting $* \times x \sim x \times *$ where $*$ is a fixed point (say $[1,0,0]$ in homogeneous coordinates, to be definite) and x is an arbitrary point in $\mathbb{C}P^2$.
(a) Compute the homomorphism $q_* : H_*(\mathbb{C}P^2 \times \mathbb{C}P^2) \to H_*(Q)$. You may give, without proof, CW structures on $\mathbb{C}P^2 \times \mathbb{C}P^2$ and Q.
(b) Find all relations among products of additive generators in the cohomology ring of Q. [Hint: you may assume q^* is injective.]

Solution: (a) Let e_0, e_2 and e_4 be the zero, two, and four-cells of one factor of $\mathbb{C}P^2$, with $e_0 = *$, and $f_0 = *$, f_2 and f_4 the cells of the second factor. $X = \mathbb{C}P^2 \times \mathbb{C}P^2$ has a product CW structure with one 0-cell - namely $e_0 \times f_0$, two 2-cells - $* \times f_2$ and $e_2 \times *$, three 4-cells - $* \times f_4$, $e_2 \times f_2$ and $e_4 \times *$, two 6-cells - $e_2 \times f_4$ and $e_4 \times f_2$, and one 8-cell, namely $e_4 \times f_4$. Because cells are all in even dimensions, they all represent homology classes which generate free abelian groups of the indicated ranks. By abuse we give the homology classes the same name.

In Q, the cells of the form $e_i \times *$ are identified with $* \times f_i$; we give the corresponding cell, and the homology class it represents, q_i. We keep the names of the other cells.

Thus we have the following homomorphisms

$$\begin{aligned}
&\dim 2 : * \times f_2 \mapsto x_2, \quad e_2 \times * \mapsto q_2 \\
&\dim 4 : * \times f_4 \mapsto q_4, \quad e_4 \times * \mapsto q_4 \quad e_2 \times f_2 \mapsto e_2 \times f_2. \\
&\dim 6 : e_4 \times f_2 \mapsto e_4 \times f_2, \quad e_2 \times f_4 \mapsto e_2 \times f_4 \\
&\dim 8 : e_4 \times f_4 \mapsto e_4 \times f_4
\end{aligned}$$

(b) We first set notation for the cohomology ring of X. Let x_2 be the Hom-dual of $e_2 \times *$ and y_2 the Hom-dual of $* \times f_2$. Then $H^* \cong \mathbb{Z}[x_2, y_2]/{x_2}^3 = 0 = {y_2}^3$. In $H^*(Q)$, the Hom-dual of q_2 which we call z_2, maps to $x_2 + y_2$. Similarly the Hom-duals of q_4 and $e_2 \times f_2$, which we denote a_4 and b_4, map to ${x_2}^2 + {y_2}^2$ and $x_2 y_2$ respectively. By naturality of cup product we compute that the images of ${z_2}^2$ and $a_4 + 2b_4$ agree in $H^*(X)$, so they are equal by our hint. The Hom-duals of $e_2 \times f_4$ and $e_4 \times f_2$, which we call c_6 and d_6, map to $x^2 y$ and xy^2. Again computing images in $H^*(X)$, $z_2 a_4 = z_2 b_4 = c_6 + d_6$. Finally, letting g_8 be the Hom-dual of $e_4 \times f_4$, maps to ${x_2}^2 {y_2}^2$. We have ${a_4}^2 = {b_4}^2 = z_2 c_6 = 2g_8$, $z_2 c_6 = z_2 d_6 = g_8$, and all other products, including $a_4 b_4 = 0$, are zero.

(8) (a) State and prove the naturality property for cap products.
(b) Let $H_c^*(X) = \lim_K H^*(X, X - K)$, where K ranges over compact sets and the maps of the directed system are the standard ones induced by inclusions at the cohain level. Let M be an oriented m-dimensional manifold. Carefully define the duality map $D_M : H_c^k(M) \to H_{m-k}(M)$ and state the Poincaré duality theorem.
(c) Deduce the standard version of the Poincaré duality theorem for oriented compact manifolds from that stated in part (b).

Solution:

(a) To state and prove naturality of cap products, the following schematic (note: it is *not* a commutative diagram) is helpful. Let $f : X \to Y$.

$$\begin{array}{ccc}
H_{n-k}(X) & \xleftarrow{\ \cap\ } & H_n(X) \times H^k(X) \\
\big\downarrow f_* & & f_* \big\downarrow \qquad f^* \big\uparrow \\
H_{n-k}(Y) & \xleftarrow{\ \cap\ } & H_n(Y) \times H^k(Y).
\end{array}$$

Let $a \in H_n(X)$ and $\phi \in H^k(Y)$. Then the naturality property for cap products is that $f_*(a) \cap \phi = f_*(a \cap f^*(\phi))$.

We prove this at the chain/cochain level. Let $\sigma : \Delta^n \to X$ be a basis element of $C_n(X)$ and let $\phi \in C^k(Y)$. Then

$$f_{\#}(\sigma) \cap \phi = \phi(f\sigma|_{[v_0, \cdots, v_k]}) f\sigma|_{[v_k, \cdots, v_n]}.$$

On the other hand,

$$\sigma \cap f^{\#}(\phi) = \phi(f\sigma|_{[v_0, \cdots, v_k]}) \sigma|_{[v_k, \cdots, v_n]}.$$

When we apply $f_\#$ to this expression, the coefficient $\phi(f\sigma|_{[v_0,\cdots,v_k]})$ remains unchanged and $\sigma|_{[v_k,\cdots,v_n]}$ becomes $f\sigma|_{[v_k,\cdots,v_n]}$ so that the expressions are equal.

(b) Let $H_*(M|K) = H_*(M, M-K)$ and similarly for cohomology. Because M is oriented, for any compact K there is an orientation class $\mu_K \in H_m(M|K)$ which restricts to the orientation generator of $H_m(M|x) \cong \mathbb{Z}$ for any $x \in K$. Moreover, if $L \subset K$, μ_K maps to μ_L under the restriction map $H_m(M|K) \to H_m(M|L)$. Because cap products of relative classes produce absolute classes we have $\cap\mu_K$ maps $H^k(M|K)$ to $H_{m-k}(M)$. Because μ_K maps to μ_L under restriction, we have that for $L \subseteq K$ the following diagram commutes

$$\begin{array}{ccc} H^k(M|L) & \xrightarrow{\cap\mu_L} & H_{m-k}(M) \\ \downarrow & & \| \\ H^k(M|K) & \xrightarrow{\cap\mu_K} & H_{m-k}(M), \end{array}$$

where the left-hand map is induced by the standard inclusion, used in the directed system over compact subsets of M defining $H^k_c(M)$. Thus the $\cap\mu_K$ define a map out from this directed system to $H_{m-k}(M)$, which gives rise to a map on the direct limit $D_M : H^k_c(M) \to H_{m-k}(M)$. The Poincareé duality theorem states that this D_M is an isomorphism.

(c) If M is compact, the directed system of compact subsets of M has a terminal element, namely M itself. The direct limit of a system indexed by a partially ordered set with a terminal object is the value at that terminal object, so $H^k_c(M) \cong H^k(M)$. Moreover, we may identify D_M with the map at the terminal object M, namely capping with μ_M, which also has the name $[M] \in H_m(M) = H_m(M|M)$. We thus deduce that for a compact, oriented M, there is a fundamental class $[M] \in H_m(M)$ such that $\cap[M] : H^k(M) \to H_{m-k}(M)$ is an isomorphism, the standard version of Poincaré duality for oriented, compact manifolds.

(9) Show that $\mathbb{C}P^2$ is not the boundary of a five-dimensional manifold.

Solution: Suppose by way of contradiction that $\mathbb{C}P^2$ were in fact ∂W, where W was a five-dimensional manifold. We construct a manifold DW as a quotient of $W \sqcup W$ by identifying their boundaries. Explicitly we take $W_{(1)} \sqcup W_{(2)}$, two copies of W, and identify $x \in \partial W_{(1)} = \mathbb{C}P^2$ with the corresponding $x \in \mathbb{C}P^2 = W_{(2)}$. We claim that DW is a five dimensional manifold. Any point away from the original $\partial W_{(i)}$ has a Euclidean neighborhood within $W_{(i)} - \partial W_{(i)}$, which is a subspace of DW. Any point originally on on the boundary of $W_{(i)}$ had a neighborhood within $W_{(i)}$ of the form $\mathbb{R}^{\geq 0} \times \mathbb{R}^4$. In DW, this point now has a neighborhood homeomorphic to the quotient of $\mathbb{R}^{\geq 0} \times \mathbb{R}^4 \sqcup \mathbb{R}^{\geq 0} \times \mathbb{R}^4$ by identifying the $0 \times \mathbb{R}^4$ subspaces, which is homeomorphic to $\mathbb{R}^5$.

A corollary of Poincaré duality is that the Euler characteristic of an odd-dimensional manifold, and thus of DW is zero, since

$$H_k(M;\mathbb{Z}/2) \cong H^{n-k}(M;\mathbb{Z}/2) \cong Hom(H_{n-k}(M;\mathbb{Z}/2),\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2),$$

and the ranks of H_k and H_{n-k} appear in the expression for $\chi(M)$ with opposite signs since n is odd. We will show that $\chi(DW) = \chi(W) + \chi(W) - \chi(\mathbb{C}P^2)$. Substituting $0 = \chi(DW)$ and reducing modulo two we would have that $\chi(\mathbb{C}P^2)$ would be even, but it is three, a contradiction which shows that DW, and thus W, cannot exist.

It remains to show that if $X = A \cup B$, which are good subspaces, then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$ when these Euler characteristics are all defined. We applied this above to $X = DW$ and A and B the images of the $W_{(i)}$. We build on our work from Problem 5.

Lemma 2. *For any exact sequence of finite-dimensional vector spaces* $0 \to A_N \xrightarrow{f_N} A_{N-1} \xrightarrow{f_{N-1}} \cdots \to A_1 \to 0$, *we have* $\Sigma_{i=1}^N \text{rank } A_i = 0$.

Indeed, any such exact sequence can be broken up into short exact sequences $0 \to \text{Im } f_{i+1} \to A_i \to \text{Im } f_i \to 0$, so by substituting rank Im f_{i+1} + rank Im f_i for the rank of A_i in the alternating sum, we get a sum where all terms cancel in pairs.

Now let H_n denote homology with rational coefficients for simplicity. The Mayer-Vietoris sequence for X reads

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots.$$

Applying the lemma we get the alternating sum of ranks in this sequence is zero, but by reordering terms we have that $\chi(A) + \chi(B) - \chi(X) - \chi(A \cap B)$ is zero, as needed.