2. Possible Quals problems from Math 638/9 for Winter 04

prob2.1 **Problem 2.1.** Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. Give V an inner-product of signature $(1,2)$ by setting

$$(e_1, e_1) = (e_2, e_2) = +1, \ (e_3, e_3) = -1, \ (e_i, e_j) = 0 \text{ for } i \neq j\,.$$

Give the pseudosphere $S = \{v = xe_1 + ye_2 + ze_3 \in V : (v, v) = -1 \text{ and } z > 0\}$ the induced pseudo-Riemannian metric. Show that S is a complete Riemannian manifold with constant non-positive scalar curvature. Justify carefully all the steps in your calculation and if you use any theorems, state them carefully.

prob2.2 **Problem 2.2.** The following is a well known theorem in Differential geometry:

thm2 **Theorem 2.** *Let f_1 and f_2 be smooth maps from $M \to N$ which are homotopic. Then $f_1^* = f_2^*$ acting on the DeRham cohomology groups.*

An essential ingredient in the proof of this result was the following

lem2 **Lemma 2.** *Let M be a smooth manifold. Let X be a smooth non-vanishing vector field on M and let ω be a smooth p form on M. Let $\mathcal{L}_X\omega$ be the Lie Derivative of X and let* $\operatorname{int}(X)$ *be interior multiplication by X. Then $\mathcal{L}_X\phi = d\operatorname{int}(X)\phi + \operatorname{int}(X)d\phi$.*

Define $\mathcal{L}_X$ and $\operatorname{int}(X)$. Give a careful proof of Lemma 2. Then use Lemma 2 to prove Theorem 2.

prob2.3 **Problem 2.3.** Let $F(\theta, \phi) := (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$ parametrize the sphere of radius 2 in $\mathbb{R}^3$. Compute the second fundamental form of this surface and use this computation to determine the scalar curvature.

prob2.4 **Problem 2.4.** Let (M, g) be a Riemannian manifold.

(1) Give a careful statement of a theorem in differential geometry that discusses various notions of 'completeness'.
(2) Prove or disprove the following assertion: "Suppose that (M, g) is a Riemannian manifold and that any two points P, Q in M can be joined by a geodesic σ of length equal to $d(P, Q)$. Then M is complete."

prob2.5 **Problem 2.5.** Show that S^3 does not admit a metric of constant sectional curvature -1.

prob2.6 **Problem 2.6.** Let S^2 be the unit sphere in $\mathbb{R}^3$. Let $\sigma(t) = (\cos t, \sin t, 0)$ be a unit speed geodesic on S^2. Let Y be a Jacobi field along σ so that $Y(0) = (0, 1, 0)$ and $Y(\frac{\pi}{2}) = (2, 0, 1)$. Determine $Y(\frac{\pi}{4})$.

prob2.7 **Problem 2.7.** Let $\mathfrak{g}_\ell$ be the Lie algebra of left invariant vector fields on a Lie group G and let $\mathfrak{g}_\ell^*$ be the set of left invariant 1 forms on S^3. Let

$$H_\ell^p(G) := \frac{\ker\{d : \Lambda^p(\mathfrak{g}_\ell^*) \to \Lambda^{p+1}(\mathfrak{g}_\ell^*)\}}{\operatorname{image}\{d : \Lambda^{p-1}(\mathfrak{g}_\ell^*) \to \Lambda^p(\mathfrak{g}_\ell^*)\}}$$

be the left invariant de Rham cohomology groups.

(1) Determine $H_\ell^p(S^3)$ for $\ell = 0, 1, 2, 3$. Cite carefully any results that you use.
(2) Cite a theorem which relates these invariant de Rham cohomology groups to the ordinary de Rham cohomology groups.

Solution Problem 2.1 Let $O(1,2)$ be the group that preserves this inner product; $O(1,2)$ then acts by isometries on $S \cup -S$. We can parametrize a point of S in the form $T(\theta,\phi) := (\cos\theta\sinh\phi, \sin\theta\sinh\phi, \cosh\phi)$. Setting then

$$g(\theta\phi) := \begin{pmatrix} -\sin\theta & \cos\theta\cosh\phi & \cos\theta\sinh\phi \\ \cos\theta & \sin\theta\cosh\phi & \sin\theta\sinh\phi \\ 0 & \sinh\phi & \cosh\phi \end{pmatrix}$$

then defines an element of $O(1,2)$ so $g(\theta,\phi)e_3 = T(\theta,\phi)$. Thus $O(1,2)$ acts transitively on S by isometries. The matrix $diag(1,1,-1)$ interchanges the two components. Thus S is a homogeneous space. We showed in class that any homogeneous space is complete and has constant scalar curvature. If S had positive scalar curvature, then S would be compact. It is not. Discussed in class.

Solution Problem 2.2: Let ϕ_t^X be the flow – this is defined in a neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. Then $\mathcal{L}_X\omega = \lim_{t\to 0}\frac{(\phi_t^X)^*\omega-\omega}{t}$. Since ω is a p form, it is a totally anti-symmetric bilinear form. Thus we may define a totally anti-symmetric $p-1$ form $\text{int}(X)$ by setting $\text{int}(X)\omega[X_2,...,X_p] = \omega(X,X_2,...,X_p)$. As X is non-vanishing, we may choose local coordinates $x = (x_1,...,x_m)$ so that $X = \partial_1^x$ and $\Phi_t^X : P \to P + te_1$ where $e_1 = (1,0,...,0)$. Let $I = (i_1,...,i_p)$ be a collection of indices with $2 \le i_1 < ... < i_p \le m$ and let $dy^I = dx^{i_1}\wedge ... \wedge dx^{i_p}$. We may expand $\omega = \alpha_I dy^I + \beta_J dx^1 \wedge dy^J$. We then have

$$\begin{aligned}
&\mathcal{L}_X\omega = \partial_1^x\alpha_I dy^I + \partial_1^x\beta_J dx^1\wedge dy^J,\\
&\text{int}(X)\omega = \beta_J dy^J,\\
&d\,\text{int}(X)\omega = \partial_1\beta_J dx^1\wedge dy^J + \textstyle\sum_{k>1}\partial_k^x\beta_J dx^k\wedge dy^J,\\
&d\omega = \partial_1^x\alpha_I dx^1\wedge dy^I + \textstyle\sum_{k>1}\partial_k^x\phi_I dx^k\wedge dy^I - \sum_{k>1}\partial_k^x\beta_J dx^1\wedge dy^k\wedge dy^J,\\
&\text{int}(X)d\omega = \partial_1^x\alpha_I dy^I - \textstyle\sum_{k>1}\partial_k^x\beta_J\wedge dy^J\,.
\end{aligned}$$

from which the desired formula follows.

Let ω be a closed p form on N. Then using the naturality of the exterior derivative, we may compute

$$\begin{aligned}
\phi_1^*\omega - \phi_0^*\omega &= \textstyle\int_{t=0}^1 i_t^*\mathcal{L}_{\partial_t}\Phi^*\omega dt = \int_{t=0}^1 i_t^*\{dint(\partial_t) + int(\partial_t)d\}\Phi^*\omega dt\\
&=.\ d\textstyle\int_{t=0}^1 i_t^* int(\partial_t)\Phi^*\omega dt\,.
\end{aligned}$$

This shows $[\phi_1^*\omega] - [\phi_0^*\omega] = 0$ in de Rham cohomology as desired.

Solution Problem 2.3 Since $F(\theta,\phi) := (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$, The normal vector is given by setting $\nu = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Thus, abusing notation slightly,

$$\begin{aligned}
L_{\theta\theta} &= \tfrac{\partial^2 F}{\partial\theta^2}\cdot\nu = 2(-\cos\theta\sin\phi, \sin\theta\sin\phi, 0)\cdot(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) = -2\sin\phi^2,\\
L_{\theta\phi} &= \tfrac{\partial^2 F}{\partial\theta\partial\phi}\cdot\nu = 2(-\sin\theta\cos\phi, \cos\theta\cos\phi, 0)\cdot(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) = 0,\\
L_{\phi\phi} &= \tfrac{\partial^2 F}{\partial\phi^2}\cdot\nu = 2(-\cos\theta\sin\phi, -\sin\theta\sin\phi, -\cos\phi)\cdot\nu = -2\,.
\end{aligned}$$

We also compute

$$\begin{aligned}
g_{\theta\theta} &= \tfrac{\partial F}{\partial\theta}\cdot\tfrac{\partial F}{\partial\theta} = 4(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)\cdot(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) = 4\sin^2\phi,\\
g_{\theta\phi} &= \tfrac{\partial F}{\partial\theta}\cdot\tfrac{\partial F}{\partial\phi} = 4(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)\cdot(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) = 0,\\
g_{\phi\phi} &= \tfrac{\partial F}{\partial\phi}\cdot\tfrac{\partial F}{\partial\phi} = 4(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\cdot(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) = 4,
\end{aligned}$$

We may now see that the scalar curvature is given by

$$K = \frac{L_{\theta\theta}L_{\phi\phi} - L_{\theta\phi}^2}{g_{\theta\theta}g_{\phi\phi} - g_{\theta\phi}^2} = \frac{1}{4}\,.$$

This works for $0 < \phi < \pi$, $0 \le \theta \le 2\pi$. And by continuity it also holds at the north and south poles.

Solution Problem 2.4 This is Hopf-Rinow's theorem:

Theorem 4. *Let (M,g) be a connected Riemannian manifold. The following assertions are equivalent:*

(1) *The geodesic distance function $d_g(P,Q) = inf_{\gamma:\gamma(0)=P,\gamma(1)=Q}L(\gamma)$ on M makes M into a complete metric space.*
(2) *There exists a point $P_0 \in M$ so that any geodesic thru P_0 extends for infinite time.*

(3) *Every geodesic on M extends for infinite time*

There is an interesting corollary. If (M, g) satisfies any of the 3 equivalent conditions described above and if P and Q are arbitrary points of M, then there exists at least 1 geodesic σ from P to Q so that $L(\sigma) = d_g(P, q)$. The converse assertion is, however false. Consider the open unit interval $(0,1) \subset \mathbb{R}$. Given any 2 points in $(0,1)$, there exists a unique straight line between them. But this is not a complete metric space.

Solution Problem 2.5 If S^3 admits a metric of constant negative sectional curvature -1, then the universal cover of S^3 is diffeomorphic to $\mathbb{R}^3$. On the other hand, S^3 is simply connected. Thus this would imply that S^3 itself is diffeomorphic to $\mathbb{R}^3$. Since S^3 is compact and $\mathbb{R}^3$ is not compact, this is not possible.

Solution Problem 2.6 Let σ be a unit speed geodesic in S^2. Let $\{e_1, e_2\}$ be a parallel frame for TS^2 along $\dot{\sigma}$ so that $e_1 = \dot{\sigma}$. Then $(a + bt)e_1$ is a Jacobi vector field along σ. Since S^2 has constant sectional curvature, $(c \cos t + d \sin t)e_2$ also is a Jacobi vector field along σ. In this particular instance take $e_2 = (0, 0, 1)$. We want to have $Y(0) = e_1$ and $Y(\frac{\pi}{2}) = 2e_1 + e_2$. Well, that says that $a = 1$, $b = 1$, $c = 0$, and $d = 1$. Substituting now yields $(\frac{3}{\sqrt{2}}\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}$, modulo numerical mistakes.

Solution Problem 2.7 Let G be a matrix group. If $a, b \in T_I G$, let X_a^ℓ and X_b^ℓ be the associated left invariant vector fields. We showed

$$[X_a^\ell, X_b^\ell] = X_{[a,b]}^\ell \,.$$

Let $e_1(g) = g \cdot i$, $e_2(g) = g \cdot j$, and $e_3(g) = g \cdot k$ be the usual basis for the Lie algebra of left invariant vector fields on S^3. Let e^i be the corresponding dual basis for the space of left invariant 1 forms. We showed

$$de^i(e_j, e_k) = e_j(e^i(e_k)) - e_k(e^i(e_j)) - e^i([e_j, e_k]) = -e^i([e_j, e_k]) \,.$$

Applying this to the setting at hand then yields

$$de^1 = -2e^2 \wedge e^3, \quad de^2 = -2e^3 \wedge e^1, \quad \text{and} \quad de^3 = -2e^1 \wedge e^2 \,.$$

Thus $d : \Lambda^1(\mathfrak{g}_\ell^*) \to \Lambda^2(\mathfrak{g}_\ell^*)$ is an isomorphism. It now follows

$$H_\ell^0(S^3) = [1] \cdot \mathbb{R} \quad \text{and} \quad H_\ell^3(S^3) = [e^1 \wedge e^2 \wedge e^3] \cdot \mathbb{R}$$

whilst $H_\ell^1(S^3) = 0$ and $H_\ell^2(S^3) = 0$. We also showed that the equivariant cohomology groups were isomorphic to the ordinary DeRham cohomology groups for compact connected Lie groups.

3. An extra two problems

prob3.1 **Problem 3.1.** The Maurier-Cartan forms played an important role in our computation of $H^*_{DeR}(U(n))$.

(1) Give a careful definition of the Maurier-Cartan form $\Theta_{2k-1} \in C^\infty(\Lambda^{2k-1})(U(n))$.
(2) Show Θ_{2k-1} is closed.

prob3.2 **Problem 3.2.** The Maurier-Cartan forms played an important role in our computation of $H^*_{DeR}(U(n))$. Show that for any k that there exists $n = n(k)$ and a map $f_k : S^{2k-1} \to U(n)$ so that $f_k^* \Theta_{2k-1} \neq 0$ in $H^{2k-1}_{DeR}(S^{2k-1})$.

Solution Problem 3.1 Let g be the identity map from $U(n)$ to $M_n(\mathbb{C})$. Then $g^{-1}dg$ is a matrix of left-invariant differential forms and $\Theta_{2k-1} := \operatorname{Tr}\{(g^{-1}dg)^{2k-1}\}$. Since $d(g^{-1}) = -g^{-1}dg \cdot g^{-1}dg$, we have

$$\begin{aligned} d\Theta_{2k-1} &= = \operatorname{Tr}\{-(g^{-1}dg)^2(g^{-1}dg)^{2k-2} + g^{-1}dg\{(g^{-1}dg)^2\}(g^{-1}dg)^{2k-3} - ...) \\ &= -\operatorname{Tr}\{(g^{-1}dg)^{2k}\} \end{aligned}$$

However we may now use the cyclic nature of the trace to compute $\operatorname{Tr}(AB) = -\operatorname{Tr}(BA)$ where A is a 1 form valued matrix and B is a $2k-1$ form valued matrix. Thus

$$-\operatorname{Tr}\{(g^{-1}dg)^{2k}\} = \operatorname{Tr}\{(g^{-1}dg)^{2k-1}(g^{-1}dg)\} = \operatorname{Tr}\{(g^{-1}dg)^{2k}\}$$

which implies this vanishes as desired

Solution Problem 3.2 Suppose that $f : \mathbb{R}^{2k} \to M_n(\mathbb{C})$ is linear with $f(x)^*f(x) = |x|^2 \operatorname{id}$. Then f induces a map from S^{2k-1} to $U(n)$. Let $f(x) = x_1e_1 + ... + x_ne_n$. We then have $e_i^*e_j + e_j^*e_i = 2\operatorname{id}$. Choose any point $P \in S^{2k-1}$. Choose coordinates so $P = (1, 0, ..., 0)$. Then $f(x)^{-1}df = e_1^*(e_2dx_2 + ... + e_{2n}dx_n)$. Let $\alpha_i = e_1^*e_i$. Then $\{\alpha_2, ..., \alpha_n\}$ are Clifford matrices with $\alpha_i\alpha_j + \alpha_j\alpha_i = -2\delta_{ij}$. Thus $f^*\Theta_{2n-1} = (2n-1)!\operatorname{Tr}(\alpha_2...\alpha_{2n})dvol$. On the other hand since $(\alpha_2...\alpha_{2n})^2 = \pm\operatorname{id}$, we have $\varepsilon(P) := \operatorname{Tr}(\alpha_2...\alpha_{2n})$ belongs to $\mathbb{N}$ or to $\sqrt{-1}\mathbb{N}$. Thus in particular it takes discrete values and must be independent of P. Thus we need only construct an example where $\varepsilon(P_0) \neq 0$ for some point P_0 of S^{2n-1}. We reverse the process. Let $\{\alpha_2, ..., \alpha_{2n}\}$ be Clifford matrices with $\operatorname{Tr}(\alpha_2...\alpha_{2n}) \neq 0$. Set $f(x) = x_1 + \alpha_2x_2 + ... + \alpha_{2n}x_{2n}$.