QUALIFYING EXAM, Winter 2004
Algebraic Topology and Differential Geometry

NAME ________________________________________ (PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ___________________________ SIGNATURE ___________________________

Please do any 10 problems out of the following 20.

1. Let \( \pi \) be a finite abelian group. Compute the groups \( H^1(K(\pi,1); \mathbb{Z}) \) and \( H_1(K(\pi,1); \mathbb{Z}) \). Please explain the details.

2. Define the Whitehead product. Prove that the element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) is in the kernel of the suspension homomorphism
\[
\Sigma : \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).
\]

3. Let \( A : S^n \to S^n \) be the antipodal map, \( A : x \mapsto -x \), and \( t_n \in \pi_n(S^n) \) be the generator represented by the identity map \( S^n \to S^n \). Prove that the homotopy class \([A] \in \pi_n(S^n)\) is equal to
\[
[A] = \begin{cases} 
  t_n, & \text{if } n \text{ is odd} \\
  -t_n, & \text{if } n \text{ is even}
\end{cases}
\]

4. Let \( A \subset X \), and \((X, A)\) be a Borsuk pair (for example, a CW-pair). Let \( E = C(X, Y), B = C(A, Y) \), and the map \( p : E \to B \) be defined as \( p(f : X \to Y) = (f|_A : A \to Y) \). Prove that the map \( p : E \to B \) is a Serre fiber bundle.

5. State the Lefschetz Fixed Point Theorem. Let
\[
f : \mathbb{C}P^{2k} \vee \mathbb{R}P^{2n} \to \mathbb{C}P^{2k} \vee \mathbb{R}P^{2n}
\]
be a map. Prove that \( f \) always has a fixed point.

6. Prove the spaces \( \mathbb{R}P^n \times S^k \) and \( S^n \times \mathbb{R}P^k \) are homotopy equivalent if and only if \( k = n \).

7. Define the Hopf invariant \( h(\lambda) \) of an element \( \lambda \in \pi_{2k-1}(S^k) \). Prove that \( h([t_{2q}, t_{2q}]) = 2 \), where \( t_{2q} \in \pi_{2q}(S^{2q}) \) is the standard generator.

8. Prove that \( \mathbb{C}P^4 \) is not homotopy equivalent to \( \mathbb{C}P^3 \times S^8 \).

9. Prove that a compact closed oriented manifold \( M \) of dimension 2003 has a nowhere vanishing tangent vector field.

10. Define a cup-product in cohomology. Let \( X = \mathbb{R}P^n \times \mathbb{R}P^k \). For each prime \( p \) determine a ring structure of \( H^*(X; \mathbb{Z}/p) \).
11. Prove that the product $M \times N$ of two manifolds is orientable if and only if both $M$ and $N$ are orientable.

12. Let $\alpha \in \Omega^1(M^3)$ be an one-form on a 3-manifold $M$ such that $\alpha_p \neq 0$ for every $p \in M$. Show that the rank-two distribution $S \subset TM$ given by $S_p = \text{Ker}(\alpha_p)$ is integrable if and only if $d\alpha \wedge \alpha = 0$.

13. Let $f : M \to N$ be a submersion of smooth manifolds. Prove that if $M$ is compact and $N$ is connected, then $f$ is onto.

14. Let $\{e_1, e_2, e_3\}$ be the standard basis for $V = \mathbb{R}^3$. Give $V$ an inner-product of signature $(1, 2)$ by setting

\[(e_1, e_1) = (e_2, e_2) = +1, \quad (e_3, e_3) = -1, \quad (e_i, e_j) = 0 \text{ for } i \neq j.\]

Give the pseudosphere $S = \{v = xe_1 + ye_2 + ze_3 \in V : (v, v) = -1 \text{ and } z > 0\}$ the induced pseudo-Riemannian metric. Show that $S$ is a complete Riemannian manifold with constant non-positive scalar curvature. Justify carefully all steps in your calculation and if you use any theorems, state them carefully.

15. The following is well-known theorem in Differential geometry:

**Theorem 1.** Let $f_1, f_2 : M \to N$ and $f_2$ be smooth maps which are homotopic. Then $f_1^* = f_2^*$ acting on the DeRham cohomology groups.

An essential part in the proof of this result was the following

**Lemma 2.** Let $M$ be a smooth manifold. Let $X$ be a smooth non-vanishing vector field on $M$ and let $\varphi$ be a smooth $p$-form on $M$. Then $L_X \varphi = d(\text{int}(X)\varphi) + \text{int}(X)d\varphi$, where $L_X$ and $\text{int}(X)$ are the Lie and the inner derivatives with respect to $X$.

Give definition of $L_X$ and $\text{int}(X)$. Give a careful proof of Lemma 2. Then use Lemma 2 to prove Theorem 1.

16. Let $F(\theta, \varphi) := (2\cos \theta \sin \varphi, 2\sin \theta \sin \varphi, 2\cos \varphi)$ parametrize the sphere of radius 2 in $\mathbb{R}^3$. Compute the second fundamental form of this surface and use this computation to determine the scalar curvature.

17. Let $(M, g)$ be a Riemannian manifold.

(a) Give a careful statement of a theorem in differential geometry that discusses various notions of 'completeness'.

(b) Prove or disprove the following assertion: "Suppose that $(M, g)$ is a Riemannian manifold and that any two points $P, Q$ in $M$ can be joined by a geodesic $\sigma$ of length equal to $d(P, Q)$. Then $M$ is complete."

18. Show that $S^3$ does not admit a metric of constant sectional curvature $-1$.

19. Let $S^2$ be the unit sphere in $\mathbb{R}^3$. Let $\sigma(t) = (\cos t, \sin t, 0)$ be a unit speed geodesic on $S^2$. Let $Y$ be a Jacobi field along $\sigma$ so that $Y(0) = (0, 10)$ and $Y\left(\frac{\pi}{2}\right) = (2, 0, 1)$. Determine $Y\left(\frac{\pi}{2}\right)$.

20.

(a) Let $G$ be a Lie group and let $V = T_eG$ be the tangent space at the identity. Prove that for every vector $v \in V$ there exists a unique left invariant vector field $X_v$ on $G$ such that $X_v(e) = v$.

(b) For $G = SU(2)$, the group of $2 \times 2$ unitary matrices with determinant one, choose a basis $f_1, f_2, f_3 \in T_eG$ and compute the commutators of the left-invariant vector fields $X_{f_1}, X_{f_2}, X_{f_3}$ corresponding to $f_1, f_2, f_3$. 
1a) The R³ representations of the tangent vectors at a point \( \left( \frac{2}{e^{xy}} \right) \) are \( \frac{dx}{ds} = \left( \frac{1}{e^{xy}} \right) \) and \( \frac{dy}{ds} = \left( \frac{0}{e^{xy}} \right) \). So at the point \( \left( \frac{1}{e^2} \right) \), the tangent plane is given by

\[
P(s,t) = \left( \frac{1}{e^2} \right) + s \frac{1}{e^{xy}} + t \frac{0}{e^{xy}}
\]

\[
= \left( \frac{1}{e^2} \right) + s \left( \frac{1}{e^2} \right) + t \left( \frac{0}{e^2} \right)
\]

b) The components of the first fundamental form are given by

\[
\begin{align*}
g_{xx}(\frac{2}{e^{xy}}, \frac{2}{e^{xy}}) &= \left< \frac{2}{e^{xy}}, \frac{2}{e^{xy}} \right> \\ &= \left< \left( \frac{1}{e^{xy}} \right), \left( \frac{1}{e^{xy}} \right) \right> \\ &= 1 + e^{2(xy)} \\
g_{yy}(\frac{2}{e^{xy}}, \frac{2}{e^{xy}}) &= \left( \frac{2}{e^{xy}}, \frac{2}{e^{xy}} \right) \\
&= e^{2(xy)}
\end{align*}
\]

\[
\begin{align*}
g_{xy}(\frac{2}{e^{xy}}, \frac{2}{e^{xy}}) &= \left< \frac{0}{e^{xy}}, \frac{0}{e^{xy}} \right> \\ &= \left< \left( \frac{0}{e^{xy}} \right), \left( \frac{0}{e^{xy}} \right) \right> \\ &= 1 + e^{2(xy)}
\end{align*}
\]

\[
\begin{align*}
\text{Area} &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sqrt{1 + 2e^{2(xy)}} \, dx \, dy \\
\text{But det } \mathbf{g} &= \det \begin{pmatrix} 1 + e^{2(xy)} & e^{2(xy)} \\ e^{2(xy)} & 1 + e^{2(xy)} \end{pmatrix} = 1 + 2e^{2(xy)} + e^{4(xy)} - e^{4(xy)} = 1 + 2e^{2(xy)} \\
\text{So } \text{Area} &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sqrt{1 + 2e^{2(xy)}} \, dx \, dy
\end{align*}
\]
d) Length \([\alpha] = \int_0^1 \sqrt{g(\beta', \beta')} \, dt\)

\[
= \int_0^1 \sqrt{g_{xx}(\beta) \left(\frac{dx}{dt}\right)^2 + 2g_{xy}(\beta) \frac{dx}{dt} \frac{dy}{dt} + g_{yy}(\beta) \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_0^1 \sqrt{(1 + e^{2t + 2t^2})(1) + (1 + e^{2t + 2t^2})(2t^2) + 2e^{2t + 2t^2}(2t)} \, dt
\]

\[
= \int_0^1 \sqrt{1 + 4t^2 + e^{2t + 2t^2}} \left(1 + 4t^2 + 4t\right) \, dt
\]
b) $K$ is an $(0,2)$ tensor so long as it is linear. We calculate

$$K(\alpha \nu + w, u) = -\langle D_\alpha \nu + w, e_+ \rangle$$

$$= -\langle \alpha D_\nu e_+ + D_w e_+, u \rangle$$

$$= -\alpha \langle D_\nu e_+, u \rangle - \langle D_w e_+, u \rangle$$

$$= \alpha K(\nu, u) + K(w, u)$$

$$K(\nu, \alpha w + u) = -\langle D_\nu e_+, \alpha w + u \rangle$$

$$= -\alpha \langle D_\nu e_+, w \rangle - \langle D_\nu e_+, u \rangle$$

$$= +\alpha K(\nu, w) + K(\nu, u)$$

c) Since $K$ is a tensor, it suffices to show that

$$K(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = K(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$$

LHS = $K(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$

$$= -\langle D_\nu e_+, \frac{\partial}{\partial y} \rangle$$

$$= \frac{\partial}{\partial x} \langle -e_+, \frac{\partial}{\partial y} \rangle + \langle e_+, \frac{\partial}{\partial x} \frac{\partial}{\partial y} \rangle$$

$$= 0 + \langle e_+, \frac{\partial}{\partial y} \text{O} \rangle$$

where $\text{O}$ represents the surface

$$= \langle e_+, \frac{\partial}{\partial y} \text{O} \rangle$$

Similarly,

RHS = $\langle e_+, \frac{\partial}{\partial \frac{\partial}{\partial y}} \rangle$ \nu

d) In this case, there must be an extra slot for the normal vector field, which now comes from the 2-dimensional space of normal vectors to the surface.
3) A subset $\Sigma$ of $M$ is an embedded submanifold of $M$ if for each $p \in \Sigma \cap M$, there exists a chart $(U, \phi)$ of $M$ such that $\Sigma \cap U$ is a $k$-slice of $U$ for some natural number $k$. A subset $S \subset U$ is a $k$-slice of $U$ if
\[ S = \{ (x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \mid x^{k+1}, \ldots, x^n \text{ are constants} \} \]

b) To show that $SL(n, \mathbb{R})$ is a submanifold of $GL(n, \mathbb{R})$, rely on Thm: If $\Phi: M \to N$ is smooth and has constant rank, then every nonempty level set $\Phi^{-1}(a)$ is a closed embedded submanifold (Corollary of Implicit Func Thm).

Since $det$ is smooth, since $SL(n, \mathbb{R}) = det^{-1}(1)$, and since $det$ has constant rank, result follows.

c) Similar argument, relying on the map $S$, i.e. its constant rank.
\[ a> i^* g = dx^2 + dy^2 + dz^2 + (d(x^2 + y^2 + z^2))^2 \\
= dx^2 + dy^2 + dz^2 + (2x dx + 2y dy + 2z dz)^2 \\
= (1 + 4x^2) dx^2 + (1 + 4y^2) dy^2 + (1 + 4z^2) dz^2 \\
+ 8xy dx dy + 8xz dx dz + 8yz dy dz \\
\]

To argue that this is Riemannian, one may check positive definiteness, or use the fact that \( i(R^3) \) is an embedded submanifold of \( H^4 \).

\[ b> i^* \omega = dx \wedge dz + dy \wedge (2x dx + 2y dy + 2z dz) \\
= dx \wedge dz + 2y dy \wedge dx + 2z dy \wedge dz \\
\]

This is not a symplectic 2-form, since it lives on a three dimensional manifold.

\[ c> \text{The volume is given by} \\
\int \sqrt{\det i^* g} \, dx \wedge dy \wedge dz = \int \sqrt{1 + 4x^2 + 4y^2 + 4z^2 + 12x^2 y^2 + 12x^2 z^2 + 12y^2 z^2} \, dx \wedge dy \wedge dz, \quad x^2 + y^2 + z^2 \leq 1 \\
\]

\[ d> \text{We calculate} \\
\frac{d}{dt} (t^* \omega) = \frac{d}{dt} \left[ \frac{dx}{t} \frac{dy}{t} \frac{dz}{t} \right] + \frac{d}{dt} \left[ \frac{d}{dx} \frac{d}{dy} \frac{d}{dz} \right] \\
\text{But} \quad \frac{d}{dt} x = \omega^{-1} (dH, \cdot) \\
\frac{d}{dt} x = \left( \frac{dx}{dx} \frac{dx}{dZ} + \frac{dy}{dy} \frac{dy}{dW} \right) dH \\
= \left( \frac{2H}{dx} \frac{2}{dZ} - \frac{2H}{2x} \frac{2}{dY} + \frac{2H}{dY} \frac{2}{dW} - \frac{2H}{dW} \frac{2}{dY} \right)
\]

So \[ \frac{d}{dt} (t^* \omega) = \frac{d}{dx} \left( \frac{2H}{dZ} \right) + \frac{d}{dy} \left( \frac{2H}{dY} \right) + \frac{d}{dZ} \left( \frac{2H}{dX} \right) + \frac{d}{dW} \left( \frac{2H}{dW} \right) \\
= 0 \]
5. a) So long as \( V(p) \neq 0 \), one may choose \( \alpha \) so that \( V = \frac{\partial}{\partial x^1} \)

in a neighborhood \( U \ni \mathbf{p} \). The flow \( \theta_t \) of \( V \) then takes the form

\[
\theta_t (x^1, \ldots, x^n) = (x^1 + t, \ldots, x^n)
\]

which implies that

\[
(\theta_t)_* \frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^3}
\]

We calculate

\[
(\theta_t)_* \frac{\partial}{\partial x^3} \bigg|_{t=0} = \frac{\partial}{\partial x^3} \left( W^3 (x^1 + h, \ldots, x^n) \right) \frac{\partial}{\partial x^3} 
\]

Therefore

\[
\frac{\partial}{\partial x^i} \left( W^3 (x^1 + h, \ldots, x^n) \right) \frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^i} W^3 (x^1, \ldots, x^n) \frac{\partial}{\partial x^3}
\]

Compare

\[
[L_N, W] = \left[ \frac{\partial}{\partial x^i}, W^3 \frac{\partial}{\partial x^3} \right] = \frac{\partial}{\partial x^i} W^3 \frac{\partial}{\partial x^3} \quad \text{which is the same.}
\]

b) Calculate

\[
\nabla^a W = [N, W] = \left[ \frac{\partial}{\partial x^a}, W^3 \frac{\partial}{\partial x^3} \right]
\]

\[
= \left( \frac{\partial}{\partial x^a} W^3 \right) \frac{\partial}{\partial x^b} - \left( \frac{\partial}{\partial x^b} N^a \right) \frac{\partial}{\partial x^3}
\]

\[
= \left( \frac{\partial}{\partial x^a} W^3 \right) \frac{\partial}{\partial x^b} - \left( \frac{\partial}{\partial x^b} N^a \right) \frac{\partial}{\partial x^3}
\]

\[
= \left( \frac{\partial}{\partial x^a} W^3 \right) \frac{\partial}{\partial x^b} - \left( \frac{\partial}{\partial x^b} N^a \right) \frac{\partial}{\partial x^3} = \nabla_a W^3 \frac{\partial}{\partial x^b} - \nabla_b W^3 \frac{\partial}{\partial x^3}
\]

To define \( \nabla_{\xi} \), use

\[
\nabla_{\xi} (\alpha(W)) = (\xi \nabla) (\alpha(W)) + \alpha (\xi \cdot W)
\]

We obtain them in coord form

\[
\nabla_{\xi} (\alpha \cdot \xi^k) = \left( W^3 \frac{\partial}{\partial x^i} \alpha \cdot \xi^k \right) \frac{\partial}{\partial x^k} + \left( \alpha \frac{\partial}{\partial x^k} N^m \right) \delta^k_m
\]
\[ \int_{\omega} \, d\omega + d \int_{\Omega} \omega = i_{\nabla} \frac{\omega}{\partial \alpha} \cdot \dot{\omega} (\alpha_k \, d\alpha^k) + d(\nabla_k \alpha_k) \]

\[ = i_{\nabla} \frac{\omega}{\partial \alpha} \left( \frac{\partial \alpha_k}{\partial \alpha^s} \, d\alpha^s \right) \dot{\omega} (\alpha_k \, d\alpha^k) + \left( \frac{\partial}{\partial \alpha^k} \nabla_k \right) \alpha_k \, d\alpha^k \]

\[ + \nabla^k \frac{2}{\partial \alpha^p} \alpha_k \, d\alpha^p \]

\[ = \frac{\partial \omega_k}{\partial \alpha^s} \, d\alpha^k - \frac{\partial \omega_k}{\partial \alpha^s} \, \nabla_k \, d\alpha^s \]

\[ + \frac{\partial \nabla^k}{\partial \alpha^p} \alpha_k \, d\alpha^p + \nabla^k \frac{2}{\partial \alpha^p} \alpha_k \, d\alpha^p \]

\[ = \left( \nabla^s \frac{2}{\partial \alpha^k} \alpha_k + \frac{\partial \nabla^m}{\partial \alpha^k} \alpha_m \right) \, d\alpha^k \]

The same.
6. Have \( g = e^{2f(x,y)} (dx^2 + dy^2) \)

Calculate

\[
\Gamma^x_{xx} = \frac{1}{2} g^{x^2} \left( \frac{\partial}{\partial x} g_{xx} + \frac{\partial}{\partial x} g_{mx} - \frac{1}{2} \frac{\partial^2}{\partial x^2} g_{xx} \right) \\
= \frac{1}{2} g^{xx} \left( \frac{\partial^2}{\partial x^2} g_{xx} - \frac{1}{2} \frac{\partial}{\partial x} g_{xx} \right) \\
= \frac{1}{2} e^{-2f} \left( \frac{\partial}{\partial x} g_{xx} \right) e^{2f} \\
= \frac{\partial}{\partial x} \frac{1}{2} f
\]

Similarly,

\[
\Gamma^y_{yy} = \frac{\partial}{\partial y} \frac{1}{2} f
\]

Also

\[
\Gamma^x_{xy} = \frac{1}{2} g^{x^2} \left( \frac{\partial}{\partial y} g_{mx} + \frac{\partial}{\partial x} g_{yx} - \frac{1}{2} \frac{\partial^2}{\partial x \partial y} g_{xy} \right) \\
= \frac{1}{2} g^{xx} \left( \frac{\partial}{\partial y} g_{xy} + \frac{\partial}{\partial x} g_{yx} - \frac{1}{2} \frac{\partial}{\partial x} g_{xy} \right) \\
= \frac{1}{2} e^{-2f} \left( \frac{\partial}{\partial y} g_{xy} \right) e^{2f} \\
= \frac{\partial}{\partial y} \frac{1}{2} f
\]

Similarly,

\[
\Gamma^y_{yx} = \frac{\partial}{\partial x} \frac{1}{2} f
\]

Also

\[
\Gamma^x_{yy} = \frac{1}{2} g^{x^2} \left( \frac{\partial}{\partial y} g_{my} + \frac{\partial}{\partial y} g_{ym} - \frac{1}{2} \frac{\partial^2}{\partial y^2} g_{yy} \right) \\
= \frac{1}{2} g^{xx} \left( \frac{\partial}{\partial y} g_{yy} + \frac{\partial}{\partial y} g_{ym} - \frac{1}{2} \frac{\partial}{\partial y} g_{yy} \right) \\
= \frac{1}{2} e^{-2f} \left( \frac{\partial}{\partial y} g_{yy} \right) e^{2f} \\
= \frac{\partial}{\partial y} \frac{1}{2} f
\]

Similarly,

\[
\Gamma^y_{xx} = -\frac{\partial}{\partial x} \frac{1}{2} f
\]

This completes the Christoffels.
Then for the curvature

\[ R_{x y x y} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \Gamma^y_{x x} - \frac{1}{2} \frac{\partial}{\partial y} \Gamma^y_{x x} + \Gamma^m_{y x} \Gamma^y_{x m} - \Gamma^m_{x x} \Gamma^y_{y m} \]

\[ = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{1}{2} f \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( -\frac{1}{2} f \right) + \Gamma^y_{y x} \Gamma^y_{x y} + \Gamma^y_{y x} \Gamma^y_{x x} \]

\[ - \Gamma^y_{x x} \Gamma^y_{y x} - \Gamma^y_{x x} \Gamma^y_{y y} \]

\[ = \frac{2}{\partial x^2} f + \frac{2}{\partial y^2} f + \left( \frac{1}{\partial x} f \right)^2 + \left( \frac{1}{\partial y} f \right)^2 \left( \frac{2}{\partial y} f \right) \]

\[ - \left( \frac{2}{\partial x} f \right)^2 - \left( -\frac{2}{\partial y} f \right) \left( \frac{2}{\partial y} f \right) \]

\[ = \frac{2}{\partial x^2} f + \frac{2}{\partial y^2} f \]

\[ R_{x x} = R_{m x x m} = R_{y x x y} \]

\[ = - R_{x y x y} \]

\[ = - \left( \frac{2}{\partial x^2} f + \frac{2}{\partial y^2} f \right) \]

\[ - R_{y y} = R_{x y y x} \]

\[ = - R_{x y x y} \]

\[ = - \left( \frac{2}{\partial x^2} f + \frac{2}{\partial y^2} f \right) \]

\[ R_{x y} = R_{m x y m} \]

\[ = 0 \]

\[ R = g^{x x} R_{x x} + g^{y y} R_{y y} \]

\[ = 2 e^{-2f} \left( \frac{2}{\partial x^2} f + \frac{2}{\partial y^2} f \right) \]
(7) a) Stokes' Theorem: Let \( \omega \) be a smooth \((n-1)\)-form on a smooth compact oriented \(n\)-dimensional manifold with boundary.

Then \( \int_M \omega = \int_{\partial M} \omega \).

b7) Calculate \( d\alpha \) directly:

\[
d\alpha = d \left[ \frac{xdyndz - ydxdz + zdxdy}{(x^2+y^2+z^2)^{3/2}} \right]
\]

\[
= \frac{1}{(x^2+y^2+z^2)^{3/2}} \left( dx\, dy\, nd\, dz - dy\, nd\, dz + dz\, nd\, dx \right)
+ \left( xdy\, nd\, dz - ydxdz + zdxdy \right) d \left( \frac{1}{(x^2+y^2+z^2)^{3/2}} \right)
\]

\[
= \frac{3}{(x^2+y^2+z^2)^{3/2}} \left( dx\, nd\, dz \right) - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} dx\, nd\, dz
+ \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} dx\, nd\, dz
- \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} dx\, nd\, dz
\]

\[
= 0
\]

(ii) If there were a \(1\)-form \( \beta \) such that \( d\beta = \alpha \),
then on \( \partial \Omega \),

from Stokes' theorem:

\[
\int_{\partial \Omega} \alpha = \int_{\partial \Omega} \beta \quad \text{and} \quad \int_{\partial \Omega} \beta = \int_{\partial \Omega} \beta
\]

Then accounting for orientation, determine

\[
\int_{\partial \Omega} \alpha = 0
\]

But we readily calculate \( \int_{\partial \Omega} \alpha \neq 0 \).
iii) Since $\alpha = 0$, then using the spherical annulus with inner radius $r$ and outer radius $r_2$ have

$$0 = \int \alpha = \int_0^{r_2} \int_0^{r_1} \int_{sphere of \text{ radius } r_2} \int_{sphere of \text{ radius } r_1} \alpha$$

$$= \int_0^{r_2} \int_0^{r_1} \alpha$$

Hence $\int_0^{r_2} \int_0^{r_1} \alpha$ is independent of $r$.

(iv) $\int \frac{\alpha}{(x-1)^2 + (y-2)^2 + (z-3)^2} = 0$ since the sphere is not contained in this region.

$$\int \frac{\alpha}{(x-1)^2 + (y-2)^2 + (z-3)^2} = \int \frac{\alpha}{sphere \text{ of radius } 1} = 4\pi$$

by Stokes' theorem given.
\[ a) \text{ The torsion definition gives us} \]
\[ \Gamma^a_{bc} = dx^a \left( \nabla^b_{\partial x^c} \frac{1}{2} \Gamma^d_{dc} \right) - dx^a \left( \nabla^b_{\partial x^c} \frac{1}{2} \Gamma^d_{dc} \right) = \Gamma^a_{bc} - \Gamma^a_{ob} \]

Metric compatibility tells us
\[ 0 = \partial^a_{\partial x^a} g \]
\[ = \partial^a_{\partial x^a} \left( g_{bc} dx^b dx^c \right) \]
\[ = \left( \frac{1}{2} \frac{\partial}{\partial x^a} g_{bc} - \Gamma^m_{ab} g_{mc} - \Gamma^m_{ac} g_{bm} \right) dx^b dx^c \]

\[ \Rightarrow \Gamma^m_{eb} g_{mc} + \Gamma^m_{ac} g_{bm} = \frac{1}{2} \frac{\partial}{\partial x^a} g_{bc} \]

If added, we get
\[ \frac{1}{2} \frac{\partial}{\partial x^a} g_{bc} + \frac{1}{2} \frac{\partial}{\partial x^c} g_{ba} - \frac{1}{2} \frac{\partial}{\partial x^b} g_{ac} = \Gamma^m_{ab} g_{mc} + \Gamma^m_{ac} g_{bm} \]
\[ + \Gamma^m_{eb} g_{mc} + \Gamma^m_{ca} g_{bm} \]
\[ - \Gamma^m_{bc} g_{mc} - \Gamma^m_{bc} g_{am} \]
\[ = \left( \Gamma^m_{ab} - \Gamma^m_{ba} \right) g_{mc} + \left( \Gamma^m_{cb} - \Gamma^m_{bc} \right) g_{am} \]
\[ + \Gamma^m_{ac} g_{bm} + \Gamma^m_{ca} g_{bm} \]
\[ = g^m_{ab} g_{mc} + g^m_{cb} g_{am} \]
\[ + 2 \Gamma^m_{ac} g_{bm} + Q^m_{ca} g_{bm} \]

Solve for \( \Gamma^m_{ac} \), have
\[ \Gamma^m_{ac} = \frac{1}{2} g^{mb} \left( \frac{1}{2} \frac{\partial}{\partial x^a} g_{bc} + \frac{1}{2} \frac{\partial}{\partial x^c} g_{ba} - \frac{1}{2} \frac{\partial}{\partial x^b} g_{ac} \right) \]
\[ + \frac{1}{2} g^{mb} \left( -Q^s_{ab} g_{sc} - Q^s_{ca} g_{bs} - Q^s_{cb} g_{as} \right) \]
If we use a non-coordinate basis, then we have
\[ \Gamma^m_{ab} - \Gamma^m_{ab} = Q^m_{ab} + C^m_{ab} \]

Hence we calculate, as above,
\[ \Gamma^m_{ac} = \frac{1}{2} g^{nb} \left( \frac{1}{2} \frac{\partial}{\partial x^c} g_{bc} + \frac{1}{2} \frac{\partial}{\partial x^c} g_{bs} - \frac{1}{2} \frac{\partial}{\partial x^c} g_{ac} \right) \]
\[ + \frac{1}{2} g^{nb} (-Q_{ac} g_{bc} - Q_{cs} g_{bs} - Q_{cs} g_{as}) \]
\[ + \frac{1}{2} g^{nb} (-C_{ac} g_{bc} - C_{cs} g_{bs} - C_{cs} g_{as}) \]
Q. a) (i) We need to verify transitivity of $\phi$. Let $p, q \in S^{n-1}$.

Find the great circle connecting $p$ and $q$.

Choose the axis $L$ to the circle, and choose $L$ such that $p \in L$.

We need to verify first that, if $y > 0$,

$$\text{Im} \left( \frac{a(x+iy) + b}{c(x+iy) + d} \right) > 0$$

Since $Z = \overline{W}$ for $Z, W \in \mathbb{C}$, it is sufficient to show that

$$\text{Im} \left[ (a(x+iy)+b)(c(x-iy)+d) \right] > 0$$

But $\text{Im} \left[ \right] = \text{Im} \left[ ((ax+b)+ayi)((cx+d)-cyi) \right]$

$$= (ax+b)(-cy) + ay(cx+d)$$

$$= (ad-bc)y$$

$$y > 0$$

We also need to verify that for any $(u, v), (x, y) \in \mathbb{R}^2$,

$$(a, b) \in SL(2, \mathbb{R})$$ such that $u+iv = \frac{a(x+iy) + b}{c(x+iy) + d}$

Multiply through by the denominator, take the $2 \times 2$ pieces.

have

$$ \begin{align*}
\chi a + b &= (uX - vY) + uC + ud \\
y a &= (\overline{uX} + ay) + vC + vd
\end{align*}$$

If we set $c = 0$

$$a = \sqrt{u^2} > 0$$

$$d = \gamma a = \sqrt{y^2}$$

$$b = ud - x\gamma$$

We have equality.
b) As noted above \((S^2, SO(3), \text{rotations})\) is a homogeneous space. Picking any point \(p \in S^2\), we note that the isotropy group \(SO(3)_p\) -- \(SO(3)\) rotations leaving \(p\) fixed -- is \(SO(2)\). It follows that \(\frac{SO(3)}{SO(2)}\) is diffeomorphic to \(S^2\).
Let $U \subset \mathbb{R}^n$ be a star-shaped open submanifold of $\mathbb{R}^n$. Then $H^0(U) = \mathbb{R}$ and $H^k(U) = \mathbb{R}^3$ for $k \geq 1$.

If $f \in C^\infty(U)$, we have $H^0(U) = \{ f \in C^\infty(U) \mid df = 0 \}$.

Since $U$ is connected, if $df = 0 \Leftrightarrow \frac{\partial f}{\partial x_i} = 0$ for all $i$, hence $f$ is constant. Hence $H^0(U) = \mathbb{R}$.

For $H^k(U)$ for $k \geq 1$, we first note that since $U$ is star-shaped, it is homotopy equivalent to a point $q \in U$.

But since $\mathbb{R}^3$ is a zero-dimensional manifold, we have $H^k(\mathbb{R}^3) = \mathbb{R}^3$ for $k \geq 1$.

Then by homotopy equivalence, $H^k(U) = \mathbb{R}^3$. 
