1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Winter 2004

Algebraic Topology and Differential Geometry

NAME ______________________________________

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ________________ SIGNATURE ________________

Please do any 10 problems out of the following 20.

1. Let π be a finite abelian group. Compute the groups $H^1(K(\pi,1);\mathbf{Z})$ and $H_1(K(\pi,1);\mathbf{Z})$. Please explain the details.

2. Define the Whitehead product. Prove that the element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

3. Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

4. Let $A \subset X$, and (X, A) be a Borsuk pair (for example, a CW-pair). Let $E = \mathcal{C}(X,Y)$, $B = \mathcal{C}(A,Y)$, and the map $p : E \longrightarrow B$ be defined as $p(f : X \longrightarrow Y) = (f|_A : A \longrightarrow Y)$. Prove that the map $p : E \longrightarrow B$ is a Serre fiber bundle.

5. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{2k} \vee \mathbf{RP}^{2n} \to \mathbf{CP}^{2k} \vee \mathbf{RP}^{2n}$$

be a map. Prove that f always has a fixed point.

6. Prove the spaces $\mathbf{RP}^n \times S^k$ and $S^n \times \mathbf{RP}^k$ are homotopy equivalent if and only if $k = n$.

7. Define the Hopf invariant $h(\lambda)$ of an element $\lambda \in \pi_{2k-1}(S^k)$. Prove that $h([\iota_{2q}, \iota_{2q}]) = 2$, where $\iota_{2q} \in \pi_{2q}(S^{2q})$ is the standard generator.

8. Prove that $\mathbf{CP}^4$ is not homotopy equivalent to $\mathbf{CP}^3 \times S^8$.

9. Prove that a compact closed oriented manifold M of dimension 2003 has a nowhere vanishing tangent vector field.

10. Define a cup-product in cohomology. Let $X = \mathbf{RP}^n \times \mathbf{RP}^k$. For each prime p determine a ring structure of $H^*(X;\mathbf{Z}/p)$.

11. Prove that the product $M \times N$ of two manifolds is orientable if and only if both M and N are orientable.

12. Let $\alpha \in \Omega^1(M^3)$ be an one-form on a 3-manifold M such that $\alpha_p \neq 0$ for every $p \in M$. Show that the rank-two distribution $S \subset TM$ given by $S_p = \mathrm{Ker}(\alpha_p)$ is integrable if and only if $d\alpha \wedge \alpha = 0$.

13. Let $f : M \to N$ be a submersion of smooth manifolds. Prove that if M is compact and N is connected, then f is onto.

14. Let $\{e_1, e_2, e_3\}$ be the standard basis for $V = \mathbf{R}^3$. Give V an inner-product of signature $(1,2)$ by setting

$$(e_1, e_1) = (e_2, e_2) = +1, \quad (e_3, e_3) = -1, \quad (e_i, e_j) = 0 \ \text{ for } \ i \neq j.$$

Give the pseudosphere $S = \{v = xe_1 + ye_2 + ze_3 \in V : \ (v,v) = -1 \text{ and } z > 0\}$ the induced pseudo-Riemannian metric. Show that S is a complete Riemannian manifold with constant non-positive scalar curvature. Justify carefully all steps in your calculation and if you use any theorems, state them carefully.

15. The following is well-known theorem in Differential geometry:

Theorem 1. *Let $f_1, f_2 : M \to N$ and f_2 be smooth maps which are homotopic. Then $f_1^* = f_2^*$ acting on the DeRham cohomology groups.*

An essential part in the proof of this result was the following

Lemma 2. *Let M be a smooth manifold. Let X be a smooth non-vanishing vector field on M and let φ be a smooth p-form on M. Then $\mathcal{L}_X\varphi = d(\mathrm{int}(X)\varphi) + \mathrm{int}(X)d\varphi$, where $\mathcal{L}_X$ and $\mathrm{int}(X)$ are the Lie and the inner derivatives with respect to X.*

Give definition of $\mathcal{L}_X$ and $\mathrm{int}(X)$. Give a careful proof of Lemma 2. Then use Lemma 2 to prove Theorem 1.

16. Let $F(\theta, \varphi) := (2\cos\theta \sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi)$ parametrize the sphere of radius 2 in $\mathbf{R}^3$. Compute the second fundamental form of this surface and use this computation to determine the scalar curvature.

17. Let (M, g) be a Riemannian manifold.

(a) Give a careful statement of a theorem in differential geometry that discusses various notions of 'completeness'.

(b) Prove or disprove the following assertion: "Suppose that (M, g) is a Riemannian manifold and that any two points P, Q in M can be joined by a geodesic σ of length equal to $d(P, Q)$. Then M is complete."

18. Show that S^3 does not admit a metric of constant sectional curvature -1.

19. Let S^2 be the unit sphere in $\mathbf{R}^3$. Let $\sigma(t) = (\cos t, \sin t, 0)$ be a unit speed geodesic on S^2. Let Y be a Jacobi field along σ so that $Y(0) = (0, 10)$ and $Y(\frac{\pi}{2}) = (2, 0, 1)$. Determine $Y(\frac{\pi}{4})$.

20.

(a) Let G be a Lie group and let $V = T_eG$ be the tangent space at the identity. Prove that for every vector $v \in V$ there exists a unique left invariant vector field X_v on G such that $X_v(e) = v$.

(b) For $G = SU(2)$, the group of 2×2 unitary matrices with determinant one, choose a basis $f_1, f_2, f_3 \in T_eG$ and compute the commutators of the left-invariant vector fields X_{f_1}, X_{f_2}, X_{f_3} corresponding to f_1, f_2, f_3.

Answers for Diff Geom Quals 2004

① a) The $\mathbb{R}^3$ representations of the tangent vectors at a point $\begin{pmatrix} x \\ y \\ e^{x+y} \end{pmatrix}$

are $\frac{\partial}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ e^{x+y} \end{pmatrix}$ and $\frac{\partial}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ e^{x+y} \end{pmatrix}$. So at the point $\begin{pmatrix} 1 \\ 1 \\ e^2 \end{pmatrix}$,

the tangent plane is given by

$$P(s,t) = \begin{pmatrix} 1 \\ 1 \\ e^2 \end{pmatrix} + s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}$$

$$= \begin{pmatrix} 1 \\ 1 \\ e^2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ e^2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ e^2 \end{pmatrix}$$

b) The components of the first fundamental form are given by

$$g\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\right) = \left\langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ e^{x+y} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ e^{x+y} \end{pmatrix} \right\rangle = 1 + e^{2(x+y)}$$

$$g\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = \left\langle \begin{pmatrix} 1 \\ 0 \\ e^{x+y} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ e^{x+y} \end{pmatrix} \right\rangle = e^{2(x+y)}$$

$$g\left(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}\right) = g\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = e^{2(x+y)}$$

$$g\left(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}\right) = \left\langle \begin{pmatrix} 0 \\ 1 \\ e^{x+y} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ e^{x+y} \end{pmatrix} \right\rangle = 1 + e^{2(x+y)}$$

c) $$\text{Area} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sqrt{\det g}\, dx\, dy$$

But $$\det g = \det \begin{pmatrix} 1+e^{2(x+y)} & e^{2(x+y)} \\ e^{2(x+y)} & 1+e^{2(x+y)} \end{pmatrix} = 1 + 2e^{2(x+y)} + e^{4(x+y)} - e^{4(x+y)}$$
$$= 1 + 2e^{2(x+y)}$$

So $$\text{Area} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sqrt{1+2e^{2(x+y)}}\, dx\, dy$$

d) Length $[\alpha] = \int_0^1 \sqrt{g(\beta', \beta')}\, dt$

$$= \int_0^1 \sqrt{g_{xx}(\beta)\left(\frac{dx}{dt}\right)^2 + g_{yy}(\beta)\left(\frac{dy}{dt}\right)^2 + 2g_{xy}(\beta)\frac{dx}{dt}\frac{dy}{dt}}\, dt$$

$$= \int_0^1 \sqrt{\left(1+e^{2t+2t^2}\right)(1) + \left(1+e^{2t+2t^2}\right)(2t)^2 + 2e^{2t+2t^2}(2t)}\, dt$$

$$= \int_0^1 \sqrt{1+4t^2+e^{2t+2t^2}\left(1+4t^2+4t\right)}\, dt$$

j a) $K: T_p\Sigma^2 \times T_p\Sigma^2 \to \mathbb{R}$

$v \quad w \mapsto \langle -D_v e_\perp, w\rangle$ for $e_\perp$ the normal vector at p compatible with the chosen orientation

b) K is an $\binom{0}{2}$ tensor so long as it is linear. We calculate

$$K(\alpha v + w, u) = -\langle D_{\alpha v + w} e_\perp, u\rangle$$
$$= -\langle \alpha D_v e_\perp + D_w e_\perp, u\rangle$$
$$= -\alpha\langle D_v e_\perp, u\rangle - \langle D_w e_\perp, u\rangle$$
$$= \alpha K(v,u) + K(w,u)$$

$$K(v, \alpha w + u) = -\langle D_v e_\perp, \alpha w + u\rangle$$
$$= -\alpha\langle D_v e_\perp, w\rangle - \langle D_v e_\perp, u\rangle$$
$$= +\alpha K(v,w) + K(v,u)$$

c) Since K is a tensor, it suffices to show that

$$K\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = K\left(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}\right)$$

$$LHS = K\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right)$$
$$= \left\langle -D_{\frac{\partial}{\partial x}} e_\perp, \tfrac{\partial}{\partial y}\right\rangle$$
$$= D_{\frac{\partial}{\partial x}}\left\langle -e_\perp, \tfrac{\partial}{\partial y}\right\rangle + \left\langle e_\perp, D_{\frac{\partial}{\partial x}} \tfrac{\partial}{\partial y}\right\rangle$$
$$= 0 + \left\langle e_\perp, \tfrac{\partial}{\partial x}\tfrac{\partial}{\partial y}\sigma\right\rangle \quad \text{where } \sigma \text{ represents the surface}$$
$$= \left\langle e_\perp, \tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\sigma\right\rangle$$

Similarly

$$RHS = \left\langle e_\perp, \tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\sigma\right\rangle \quad \checkmark$$

d) In this case, there must be an extra slot for the normal vector field, which now comes from the 2dim space of normal vectors to the surface

③ a) A subset Σ of M is an embedded submanifold of M if, for each $p \in \Sigma \subset M$, $\exists$ chart $(U_\alpha, \varphi_\alpha)$ of M such that $\Sigma \cap U_\alpha$ is a <u>k-slice</u> of U_α for some natural number k.
A subset $S \subset U$ is a k-slice of U if

$$S = \{(x^1, \dots x^k; x^{k+1}, \dots x^n \mid x^{k+1}, \dots x^n \text{ are constants}\}$$

b) To show that $SL(n,\mathbb{R})$ is a submanifold of $GL(n,\mathbb{R})$ rely on <u>Thm</u>: If $\varphi: M \to N$ is smooth & has constant rank, then every nonempty level set $\varphi^{-1}(c)$ is a closed embedded submanifold (corollary of Implicit Fct Thm).
Since det is smooth, since $SL(n,\mathbb{R}) = \det^{-1}(1)$, and since $D\det$ has constant rank 1, result follows

c) Similar argument, relying on the map S, & its constant rank

a) $$i^*g = dx^2+dy^2+dz^2+\left(d(x^2+y^2+z^2)\right)^2$$
$$= dx^2+dy^2+dz^2+(2x\,dx+2y\,dy+2z\,dz)^2$$
$$=(1+4x^2)\,dx^2+(1+4y^2)\,dy^2+(1+4z^2)\,dz^2$$
$$+8xy\,dx\,dy+8xz\,dx\,dz+8yz\,dy\,dz$$

To argue that this is Riemannian, one may check positive definiteness, or use the fact that $i(\mathbb{R}^3)$ is an embedded submanifold of $\mathbb{R}^4$

b) $$i^*\Omega = dx\wedge dz + dy\wedge(2x\,dx+2y\,dy+2z\,dz)$$
$$= dx\wedge dz + 2x\,dy\wedge dx + 2z\,dy\wedge dz$$

This is not a symplectic 2-form, since it lives on a three dimensional manifold

c) The volume is given by
$$\int_{x^2+y^2+z^2\le 1}\sqrt{\det i^*g}\,dx\,dy\,dz = \int\sqrt{1+4x^2+4y^2+4z^2+12x^2y^2+12x^2z^2+12y^2z^2}\,dx\,dy\,dz$$

d) We calculate
$$\frac{d}{dt}(H\circ\gamma) = \frac{\partial H}{\partial x}\frac{d}{dt}\gamma_x + \frac{\partial H}{\partial y}\frac{d}{dt}\gamma_y + \frac{\partial H}{\partial z}\frac{d}{dt}\gamma_z + \frac{\partial H}{\partial w}\frac{d}{dt}\gamma_w$$

But $\frac{d}{dt}\gamma = \omega^{-1}(dH,\ )$
$$=\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial w}\right)dH$$
$$=\frac{\partial H}{\partial x}\frac{\partial}{\partial z}-\frac{\partial H}{\partial z}\frac{\partial}{\partial x}+\frac{\partial H}{\partial y}\frac{\partial}{\partial w}-\frac{\partial H}{\partial w}\frac{\partial}{\partial y}$$

So $$\frac{d}{dt}(H\circ\gamma) = \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial z}\right)+\frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial w}\right)+\frac{\partial H}{\partial z}\left(\frac{\partial H}{\partial x}\right)+\frac{\partial H}{\partial w}\left(\frac{\partial H}{\partial y}\right)$$
$$=0$$

(5) a) So long as $V(p) \neq 0$, one may choose coordinates so that $V = \frac{\partial}{\partial x^1}$ in a neighborhood $U \ni p$. The flow θ_t of V then takes the form

$$\theta_t(x^1, \dots x^n) = (x^1 + t, \dots x^n)$$

which implies that

$$(\theta_t)_* \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}$$

We calculate

$$(\theta_{-h})_* W_{\theta_h(p)} = (\theta_{-h})_* \left[W^j(x^1+h, \dots x^n) \frac{\partial}{\partial x^j} \right]$$
$$= W^j(x^1+h, \dots, x^n) \frac{\partial}{\partial x^j}$$

Therefore

$$\mathcal{L}_V W(p) = \frac{d}{dh} (\theta_{-h})_* W_{\theta_h(p)} \Big|_{h=0}$$
$$= \frac{d}{dh} W^j(x^1+h, \dots x^n) \Big|_{h=0} \frac{\partial}{\partial x^j}$$
$$= \frac{\partial}{\partial x^1} W^j(x^1, \dots, x^n) \frac{\partial}{\partial x^j}$$

Compare

$$[V, W] = \left[\frac{\partial}{\partial x^1}, W^j \frac{\partial}{\partial x^j} \right]$$
$$= \frac{\partial}{\partial x^1} W^j \frac{\partial}{\partial x^j}$$ which is the same.

b) Calculate

$$\mathcal{L}_v w = [v, w] = \left[v^a \frac{\partial}{\partial x^a}, w^b \frac{\partial}{\partial x^b} \right]$$
$$= \left(v^a \frac{\partial}{\partial x^a} w^b \right) \frac{\partial}{\partial x^b} - \left(w^b \frac{\partial}{\partial x^b} v^a \right) \frac{\partial}{\partial x^a}$$
$$= \left(v^a \frac{\partial}{\partial x^a} w^b - w^a \frac{\partial}{\partial x^a} v^b \right) \frac{\partial}{\partial x^b}$$

c) To define $\mathcal{L}_v \alpha$, use

$$\mathcal{L}_v (\alpha(w)) = (\mathcal{L}_v \alpha)(w) + \alpha(\mathcal{L}_v w)$$

We obtain then, in coord form

$$\mathcal{L}_v (\alpha_k dx^k) = \left(v^b \frac{\partial}{\partial x^b} \alpha_k \right) dx^k + \left(\alpha_m \frac{\partial}{\partial x^k} v^m \right) dx^k$$

Compare

$$i_v d\alpha + d\, i_v \alpha = i_{v^m \frac{\partial}{\partial x^m}} d(\alpha_k dx^k) + d(v^k \alpha_k)$$

$$= i_{v^m \frac{\partial}{\partial x^m}} \left(\frac{\partial \alpha_k}{\partial x^s} dx^s \wedge dx^k \right) + \left(\frac{\partial}{\partial x^p} v^k \right) \alpha_k \, dx^p$$

$$+ v^k \frac{\partial}{\partial x^p} \alpha_k \, dx^p$$

$$= \frac{\partial \alpha_k}{\partial x^s} v^s dx^k - \frac{\partial \alpha_k}{\partial x^s} v^k dx^s$$

$$+ \frac{\partial v^k}{\partial x^p} \alpha_k \, dx^p + v^k \frac{\partial}{\partial x^p} \alpha_k \, dx^p$$

$$= \left(v^s \frac{\partial}{\partial x^s} \alpha_k + \frac{\partial v^m}{\partial x^k} \alpha_m \right) dx^k$$

The same

⑥ Have $g = e^{2f(x,y)}(dx^2+dy^2)$

Calculate

$$\Gamma^x{}_{xx} = \frac{1}{2} g^{xm}\left(\frac{\partial}{\partial x} g_{mx} + \frac{\partial}{\partial x} g_{mx} - \frac{\partial}{\partial z^m} g_{xx}\right)$$
$$= \frac{1}{2} g^{xx} \frac{\partial}{\partial x} g_{xx}$$
$$= \frac{1}{2} e^{-2f}\left(2\frac{\partial}{\partial x} f\right) e^{2f}$$
$$= \frac{\partial}{\partial x} f$$

Similarly

$$\Gamma^y{}_{yy} = \frac{\partial}{\partial y} f$$

Also $$\Gamma^x{}_{xy} = \frac{1}{2} g^{xm}\left(\frac{\partial}{\partial x} g_{my} + \frac{\partial}{\partial y} g_{mx} - \frac{\partial}{\partial z^m} g_{xy}\right)$$
$$= \frac{1}{2} g^{xx}\left(\frac{\partial}{\partial x} g_{xy} + \frac{\partial}{\partial y} g_{xx} - \frac{\partial}{\partial x} g_{xy}\right)$$
$$= \frac{1}{2} e^{-2f}\left(2\frac{\partial}{\partial y} f\right) e^{2f}$$
$$= \frac{\partial}{\partial y} f$$

Similarly

$$\Gamma^y{}_{yx} = \frac{\partial}{\partial x} f$$

Also $$\Gamma^x{}_{yy} = \frac{1}{2} g^{xm}\left(\frac{\partial}{\partial y} g_{my} + \frac{\partial}{\partial y} g_{my} - \frac{\partial}{\partial z^m} g_{yy}\right)$$
$$= \frac{1}{2} g^{xx}\left(0 + 0 - \frac{\partial}{\partial x} g_{yy}\right)$$
$$= -\frac{\partial}{\partial x} f$$

Similarly

$$\Gamma^y{}_{xx} = -\frac{\partial}{\partial y} f$$

This completes the Christoffels.

Then for the curvature

$$R_{xyx}{}^{y} = \frac{\partial}{\partial x}\Gamma^{y}{}_{yx} - \frac{\partial}{\partial y}\Gamma^{y}{}_{xx} + \Gamma^{m}{}_{yx}\Gamma^{y}{}_{xm} - \Gamma^{m}{}_{xx}\Gamma^{y}{}_{ym}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f\right) - \frac{\partial}{\partial y}\left(-\frac{\partial}{\partial y}f\right) + \Gamma^{y}{}_{yx}\Gamma^{y}{}_{xy} + \Gamma^{x}{}_{yx}\Gamma^{y}{}_{xx}$$

$$- \Gamma^{x}{}_{xx}\Gamma^{y}{}_{yx} - \Gamma^{y}{}_{xx}\Gamma^{y}{}_{yy}$$

$$= \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \left(\frac{\partial}{\partial x}f\right)^2 + \left(\frac{\partial}{\partial y}f\right)\left(-\frac{\partial f}{\partial y}\right)$$

$$- \left(\frac{\partial}{\partial x}f\right)^2 - \left(-\frac{\partial}{\partial y}f\right)\left(\frac{\partial}{\partial y}f\right)$$

$$= \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f$$

$$R_{xx} = R_{mxx}{}^{m} = R_{yxx}{}^{y}$$

$$= -R_{xyx}{}^{y}$$

$$= -\left(\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f\right)$$

$$R_{yy} = R_{xyy}{}^{x}$$

$$= -R_{xyx}{}^{y}$$

$$= -\left(\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f\right)$$

$$R_{xy} = R_{mxy}{}^{m}$$

$$= 0$$

$$R = g^{xx}R_{xx} + g^{yy}R_{yy}$$

$$= 2e^{-2f}\left(\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f\right)$$

AB

(7) a) Stokes' Thm: let ω be a smooth $(n-1)$ form on a smooth compact oriented n dimensional manifold with boundary.

Then $$\int_M d\omega = \int_{\partial M} \omega$$

b)

(i) Calculate $d\alpha$ directly:

$$d\alpha = d\left[\frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{1}{(\ )^{3/2}}\left(dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy\right)$$

$$+ (x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)\; d\left(\frac{1}{(\ )^{3/2}}\right)$$

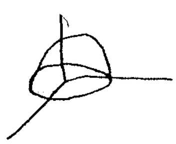
$$= (3\,dx\wedge dy\wedge dz)\frac{1}{(\ )^{3/2}} - \frac{3x^2\,dx\wedge dy\wedge dz}{(\ )^{5/2}} - \frac{3y^2\,dx\wedge dy\wedge dz}{(\ )^{5/2}}$$

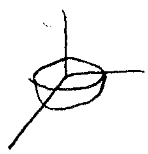
$$-\frac{3z^2\,dx\wedge dy\wedge dz}{(\ )^{5/2}}$$

$$= 0$$

(ii) If there were a 1-form β such that $d\beta = \alpha$,

then on





from Stokes':

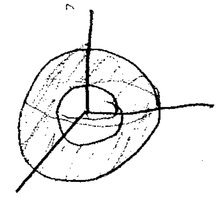
$$\int_{\ominus} \alpha = \int_{\circ} \beta \qquad \& \int_{\theta} \alpha = \int_{\circ} \beta$$

Then accounting for orientation, determine

$$\int_{\bigcirc} \alpha = 0$$

But we readily calculate $\displaystyle\int_{\ominus} \alpha \neq 0$

(iii) Since $d\alpha = 0$, then using the spherical annulus with inner radius r_1 & outer radius r_2 have



$$0 = \int_{\circledcirc} d\alpha = \int_{\substack{\text{sphere of} \\ \text{radius } r_2}} \alpha - \int_{\substack{\text{sphere of} \\ \text{radius } r_1}} \alpha$$

$\Rightarrow$ Hence $\displaystyle\int_{\text{sphere of radius } r} \alpha$ is <u>independent</u> of r.

(iv) $\displaystyle\int_{(x-1)^2+(y-2)^2+(z-3)^2=1} \alpha = 0$ since the sphere is not contained in this region

$$\int_{(x-1)^2+(y-2)^2+(z-3)^2=25} \alpha \underset{\text{by Stokes'}}{=} \int_{\substack{\text{sphere of radius 1} \\ \text{about origin}}} \alpha \underset{\text{given}}{=} 4\pi$$

8 a) The torsion definition gives us

$$Q^a{}_{bc} = dx^a\left(\nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^c}\right) - dx^a\left(\nabla_{\frac{\partial}{\partial x^c}} \frac{\partial}{\partial x^b}\right)$$
$$= \Gamma^a{}_{bc} - \Gamma^a{}_{cb}$$

Metric compatibility tells us

$$0 = \nabla_{\frac{\partial}{\partial x^a}} g$$
$$= \nabla_{\frac{\partial}{\partial x^a}} (g_{bc}\, dx^b dx^c)$$
$$= \left(\frac{\partial}{\partial x^a} g_{bc} - \Gamma^m{}_{ab} g_{mc} - \Gamma^m{}_{ac} g_{bm}\right) dx^b dx^c$$

$$\Rightarrow \Gamma^m{}_{ab} g_{mc} + \Gamma^m{}_{ac} g_{bm} = \frac{\partial}{\partial x^a} g_{bc}$$

If add, we get

$$\frac{\partial}{\partial x^a} g_{bc} + \frac{\partial}{\partial x^c} g_{ba} - \frac{\partial}{\partial x^b} g_{ac} = \Gamma^m{}_{ab} g_{mc} + \Gamma^m{}_{ac} g_{bm}$$
$$+ \Gamma^m{}_{cb} g_{ma} + \Gamma^m{}_{ca} g_{bm}$$
$$- \Gamma^m{}_{ba} g_{mc} - \Gamma^m{}_{bc} g_{am}$$
$$= (\Gamma^m{}_{ab} - \Gamma^m{}_{ba}) g_{mc} + (\Gamma^m{}_{cb} - \Gamma^m{}_{bc}) g_{ma}$$
$$+ \Gamma^m{}_{ac} g_{bm} + \Gamma^m{}_{ca} g_{bm}$$
$$= Q^m{}_{ab} g_{mc} + Q^m{}_{cb} g_{ma}$$
$$+ 2\Gamma^m{}_{ac} g_{bm} + Q^m{}_{ca} g_{bm}$$

Solve for $\Gamma^m{}_{ac}$, have

$$\Gamma^m{}_{ac} = \frac{1}{2} g^{mb}\left(\frac{\partial}{\partial x^a} g_{bc} + \frac{\partial}{\partial x^c} g_{ba} - \frac{\partial}{\partial x^b} g_{ac}\right)$$
$$+ \frac{1}{2} g^{mb}\left(-Q^s{}_{ab} g_{sc} - Q^s{}_{ca} g_{bs} - Q^s{}_{cb} g_{as}\right)$$

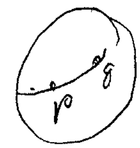
b) If we use a non coordinate basis, then we have

$$\Gamma^m{}_{ab} - \Gamma^m{}_{ab} = Q^m{}_{ab} + C^m{}_{ab}$$

Hence we calculate, as above,

$$\Gamma^m{}_{ac} = \frac{1}{2} g^{mb} \left(\frac{\partial}{\partial x^c} g_{bc} + \frac{\partial}{\partial x^c} g_{ba} - \frac{\partial}{\partial x^b} g_{ac} \right)$$
$$+ \frac{1}{2} g^{mb} \left(-Q^s{}_{ab}\, g_{sc} - Q^s{}_{ca}\, g_{bs} - Q^s{}_{cb}\, g_{as} \right)$$
$$+ \frac{1}{2} g^{mb} \left(-C^s{}_{ab}\, g_{sc} - C^s{}_{ca}\, g_{bs} - C^s{}_{cb}\, g_{as} \right)$$

(9) a) (i) We need to verify transitivity of ϕ. Let $p, q \in S^{n-1}$, & find the great circle connecting p and q. Choose the axis $\perp$ to the circle, and choose the element of $SO(n)$ which rotates about this axis, p to q.



(ii) We need to verify first that, if $y > 0$ then $\operatorname{Im}\left(\frac{a(x+iy)+b}{c(x+iy)+d}\right) > 0$

Since $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$ for $z, w \in \mathbb{C}$, it is sufficient to show that

$$\operatorname{Im}\left[(a(x+iy)+b)(c(x-iy)+d)\right] > 0$$

But

$$\begin{aligned}\operatorname{Im}[\ \ ] &= \operatorname{Im}\left[((ax+b)+ayi)((cx+d)-cyi)\right] \\ &= (ax+b)(-cy) + ay(cx+d) \\ &= (ad-bc)y \\ &= y \\ &> 0\end{aligned}$$

We also need to verify that for any $(u,v), (x,y) \in \mathbb{R}^{2,+}$, $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ such that $u+iv = \frac{a(x+iy)+b}{c(x+iy)+\phi}$

Multiply through by the denominator, take $\mathbb{R}e$ & Im pieces, have

$$xa + b = (ux - vy)c + ud$$
$$ya = (vx + uy)c + vd$$

If we set $c = 0$

$$a = \sqrt{v/y} > 0$$
$$d = 1/a = \sqrt{y/v}$$
$$b = ud - xa$$

we have equality

b> As noted above $(S^2, SO(3), \text{rotations})$ is a homogeneous space. Picking any $p \in S^2$, we note that the isotropy group $SO(3)_p$ -- $SO(3)$ rotations leaving p fixed -- is $SO(2)$. It follows that $\dfrac{SO(3)}{SO(2)}$ is diffeom to S^2

(10) Let $U \subset \mathbb{R}^n$ be a star-shaped open submanifold of $\mathbb{R}^n$. Then $H^0(U) = \mathbb{R}^1$ and $H^k(U) = \{\vec{0}\}$ $\forall\ k \geq 1$

Pf

- Have $H^0(U) = \{f \in C^\infty(U) \mid df = 0\}$
Since U is connected, if $df = 0 \Leftrightarrow \frac{\partial}{\partial x^i} f = 0$, then f is constant. Hence $H^0(U) = \mathbb{R}^1$
- For $H^k(U)$ $k \geq 1$, we first note that since U is star-shaped, it is homotopy equivalent to a point $q \in U$.
But since $\{q\}$ is a zero dimensional manifold, have $H^k(\{q\}) = \{\vec{0}\}$ for $k \geq 1$
Then by homotopy equivalence, $H^k(U) = \{\vec{0}\}$