QUALIFYING EXAM IN ALGEBRAIC TOPOLOGY WINTER 2003

You must show all your work. Carefully state any results which you use.

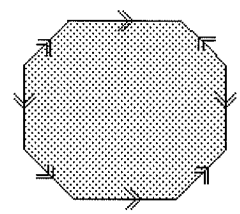
Problem #1. Show that $\pi_k(S^n, *) = 0$ for $k < n$.

Problem #2. Let $U(n)$ be the unitary group. Find $\pi_k(U(n))$ for $k = 1, 2, 3$ and $n \geq 2$.

Problem #3. Let $\rho : E \to X$ be a covering projection. Let x_0 be the basepoint of X and let $F := \rho^{-1}(x_0)$ be the fiber of ρ.

(1) Give a careful definition of the connecting homomorphism $\delta : \pi_1(X, x_0) \to F$.
(2) Show that if E is connected, then δ is surjective.

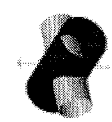
Problem #4. The sides of an octagon are glued to a circle using the pattern given to the right. Determine the fundamental group of the associated quotient space.



Problem #5. Prove or disprove the following assertion: "Let X and Y be connected finite CW complexes. Suppose that for each k there is an isomorphism ϕ_k from $\pi_k(X, x_0)$ to $\pi_k(Y, y_0)$. Then X and Y have isomorphic homology groups."

Problem #6. Determine $H_*(X)$ where

$X = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - 1)(x^2 - z^2 - .5) = 0\}$ Picture:



Problem #7. Prove or disprove the following assertion: "If X is the realization of a finite simplicial complex, then $\pi_2(X)$ is finitely generated."

Problem #8. Let X be the realization of a finite simplicial complex. Find $H_*(X; \mathbb{Z}_8)$ and $H^*(X; \mathbb{Z}_8)$ given that:

$$H_p(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 4 \\ \mathbb{Z}_3 & \text{if } p = 1, 5 \\ \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_{18} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z} & \text{if } p = 3 \end{cases}$$

Problem #9. Let f_1 and f_2 be disjoint embeddings of S^1 into $\mathbb{R}^3$. Determine the homology groups $H_*(\mathbb{R}^3 - f_1(S^1) - f_2(S^1))$

Problem #10. Let $\vec{a} = (a_1, a_2, ..., a_k)$ be a collection of positive integers sorted into increasing order $1 \leq a_1 \leq a_2 \leq ... \leq a_k$. Let $\mathbb{CP}(\vec{a}) := \mathbb{CP}^{a_1} \times ... \times \mathbb{CP}^{a_k}$ be a product of complex projective spaces. Show $\mathbb{CP}(\vec{a})$ is homotopy equivalent to $\mathbb{CP}(\vec{b})$ implies $\vec{a} = \vec{b}$.

SOLUTIONS OF QUALIFYING EXAM IN ALGEBRAIC TOPOLOGY WINTER 2003

Problem #1. Show that $\pi_k(S^n, *) = 0$ for $k < n$.

Solution Problem #1. Let $f : (S^k, *) \to (S^n, *)$ be a continuous map. The Stone-Weierstrauss theorem implies f can be uniformly approximated by a function $\tilde{f}$ which has polynomial components (and in particular differentiable). Replace $\tilde{f}$ by

$$g := \frac{* + (\tilde{f}(*) - *)}{|* + (\tilde{f}(*) - *)|}$$

to construct a differentiable map $g : (S^k, *) \to (S^n, *)$ so $|f - g|_\infty < \frac{1}{2}$. The homotopy

$$H := \frac{tf + (1-t)g}{|tf + (1-t)g|}$$

then shows $[f] = [g]$ in $\pi_k(S^n, *)$. Sard's theorem implies a differentiable map can not be surjective. Thus $g : (S^k, *) \to (S^n - P, *)$ for some point P. Let g_1 be this map. Then $[g] = i_*[g_1]$ where $i_1 : S^n - P \to S^n$ is the natural inclusion map. Since $S^n - P$ is homeomorphic to the interior of D^n and since the interior of D^n is contractible, $\pi_k(S^n - P, *) = 0$. Thus $[g_1] = 0$ so $[g] = 0$. Remark: One can substitute the simplicial approximation theorem for the use of the Stone-Weierstrauss and Sard's theorems.

Problem #2. Let $U(n)$ be the unitary group. Find $\pi_k(U(n))$ for $k = 1, 2, 3$ and $n \geq 2$.

Solution Problem #2. We showed in class that $U(2) = S^1 \times S^3$. We also showed in class that $\pi_k(X \times Y) = \pi_k(X) \oplus \pi_k(Y)$. We showed in class that $\pi_j(S^k) = 0$ for $j < k$. Thus we have $\pi_1(U(2)) = \mathbb{Z}$, $\pi_2(U(2)) = 0$, and $\pi_3(U(2)) = \mathbb{Z}$. Assume inductively that $\pi_1(U(k)) = \mathbb{Z}$, $\pi_2(U(k)) = 0$, and $\pi_3(U(k)) = 0$. We showed in class that $U(k) \to U(k+1) \to S^{2k+1}$ is a twisted product. Hence has the Covering Homotopy Property with respect to the n cube - i.e. is a Serre fibration. We now use the long exact sequence of a fibration to see:

$$\begin{aligned} &\pi_4(S^{2k+1}) \to \pi_3(U(k)) \to \pi_3(U(k+1)) \to \\ &\pi_3(S^{2k+1}) \to \pi_2(U(k)) \to \pi_2(U(k+1)) \to \\ &\pi_2(S^{2k+1}) \to \pi_1(U(k)) \to \pi_1(U(k+1)) \to \pi_1(S^{2k+1}) \end{aligned} \tag{1}$$

Since $k \geq 2$, $2k + 1 \geq 5 > 4$. Thus $\nu < 2k + 1$ for $\nu \leq 4$ and thus $\pi_\nu(S^{2k+1}) = 0$. Consequently the sequence given in equation (1) decouples and we have

$$\pi_\nu(U(k)) = \pi_\nu(U(k+1)) \text{ for } \nu = 1, 2, 3.$$

Problem #3. Let $\rho : E \to X$ be a covering projection. Let x_0 be the basepoint of X and let $F := \rho^{-1}(x_0)$ be the fiber of ρ.

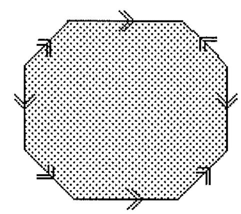
(1) Give a careful definition of the connecting homomorphism $\delta : \pi_1(X, x_0) \to F$.
(2) Show that if E is connected, then δ is surjective.

Solution Problem #3. The fundamental tool is the lifting property:

Theorem. *Let $f : (I^n, \vec{0}) \to (X, x_0)$. Let $e_0 \in F$ be the basepoint of E. Then there exists a unique $\tilde{F} : (I^n, \vec{0}) \to (E, e_0)$ so that $\rho\tilde{f} = f$.*

Let $\alpha : (I, \partial I) \to (X, x_0)$. Use the lifting property to construct $\tilde{\alpha} : (I, 0) \to (E, e_0)$ and let $\delta\alpha = \alpha(1)$; since the lift $\tilde{\alpha}$ is unique and since $\rho\tilde{\alpha}(1) = \alpha(1) = x_0$, $\delta\alpha \in F$ is well defined. Suppose $[\alpha] = [\beta]$ in $\pi_1(X, x_0)$. Then there exists a map $\Phi : I^2 \to X$ so $\Phi(0,t) = \Phi(1,t) = x_0$, $\Phi(s,0) = \alpha(s)$, and $\Phi(s,1) = \beta(s)$. Let $\tilde{\Phi}$ be the lift. Since $\rho\tilde{\Phi}(0,t) = x_0$, $\tilde{\Phi}(0,t) \in F$. Since F is discrete, $\tilde{\Phi}(0,t) = \tilde{\Phi}(0,0) = e_0$ for all t. Since $\rho\tilde{\Phi}(s,0) = \alpha(s)$ and $\rho\tilde{\Phi}(s,1) = \beta(s)$, $\tilde{\Phi}(s,0) = \tilde{\alpha}(s)$ and $\tilde{\Phi}(s,1) = \tilde{\beta}(s)$ are the unique lifts of α and β. Thus $\delta\alpha = \tilde{\Phi}(1,0)$ and $\delta\beta = \tilde{\Phi}(1,1)$. Since $\rho\tilde{\Phi}(1,t) = \Phi(1,t) = x_0$, $\tilde{\Phi}(1,t) \in F$ for all t. Since the fiber is discrete, $\tilde{\Phi}(1,0) = \tilde{\Phi}(1,1)$ and thus $\delta\alpha = \delta\beta$. Thus $\delta[\alpha] := \delta\alpha \in F$ is well defined. Since E is path connected, given any $e_1 \in F$, we can find a path $\tilde{\alpha}$ with $\tilde{\alpha}(0) = e_0$ and $\tilde{\alpha}(1) = e_1$. Let $\alpha = \rho\tilde{\alpha}$. Then $[\alpha] \in \pi_1(X, x_0)$ and $\delta[\alpha] = e_1$.

Problem #4. The sides of an octagon are glued to a circle using the pattern given to the right. Determine the fundamental group of the associated quotient space.



Solution Problem #4. One of the subcases of van Kampen's theorem that we discussed states:

Theorem. *Let $X = \mathcal{O}_1 \cup \mathcal{O}_2$ be the union of two open arc connected sets. Suppose additionally that $\mathcal{O}_1 \cap \mathcal{O}_2$ is open and that $\mathcal{O}_2$ is simply connected. Let $i_1 : \mathcal{O}_1 \cap \mathcal{O}_2 \to \mathcal{O}_1$ be the natural inclusion map. Let N be the smallest normal subgroup of $\pi_1(\mathcal{O}_1)$ which contains $(i_1)_*\pi_1(\mathcal{O}_1 \cap \mathcal{O}_2)$. Then the inclusion map j_1 of $\mathcal{O}_1$ into X induces an isomorphism from $\pi_1(\mathcal{O}_1)/N$ to $\pi_1(X)$*

With this in mind, we let $\mathcal{O}_2$ be the interior of the octagon and $\mathcal{O}_1$ be X minus the center of the octagon. This provides an admissible open cover of X. We have $\mathcal{O}_1 \cap \mathcal{O}_2$ deformation retracts to a small circle about the center and thus $\pi_1(\mathcal{O}_1 \cap \mathcal{O}_2, *) = \mathbb{Z} \cdot \alpha$ where α goes around the center once. Since $\mathcal{O}_2$ deformation retracts to the center, $\mathcal{O}_2$ is simply connected and thus we can apply the theorem cited above. We have $\mathcal{O}_1$ deformation retracts to the attaching circle so $\pi_1(\mathcal{O}_1, *) = \mathbb{Z} \cdot \beta$. Furthermore, we read around the boundary to see that:

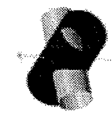
$$(i_1)_*\alpha = \beta\beta\beta^-1\beta\beta^{-1}\beta^{-1}\beta\beta = \beta^2.$$

Thus the fundamental group is $\mathbb{Z}_2$.

Problem #5. Prove or disprove the following assertion: "Let X and Y be connected finite CW complexes. Suppose that for each k there is an isomorphism ϕ_k from $\pi_k(X, x_0)$ to $\pi_k(Y, y_0)$. Then X and Y have isomorphic homology groups."

Solution Problem #5. This is false. Let $X = S^1 \vee S^2$ and $Y = S^2 \vee S^1 \vee S^2$. By van Kampen, $\pi_1(X) = \mathbb{Z}$ and $\pi_1(Y) = \mathbb{Z}$. There is a double cover $\rho : Y \to X$ obtained by wrapping the central circle around twice. Thus $\rho_* : \pi_k(Y) \to \pi_k(X)$ is an isomorphism for $k \geq 2$. Thus both spaces have isomorphic homotopy groups. However $H_2(X) = \mathbb{Z}$ and $H_2(Y) = \mathbb{Z} \oplus \mathbb{Z}$ so the spaces are not homotopy equivalent.

Problem #6. Determine $H_*(X)$ where

$X = \{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2-1)(x^2-z^2-.5) = 0\}$ Picture: 

Solution Problem #6. Let X be the intersection of a big and a medium size cylinder. We fatten things up to define an open cover $\mathcal{O}_1$ and $\mathcal{O}_2$ so $\mathcal{O}_1 \downarrow S^1 \times \mathbb{R}$, $\mathcal{O}_2 \downarrow S^1 \times \mathbb{R}$, and $\mathcal{O}_1 \cap \mathcal{O}_2 \downarrow S^1 \dot\sqcup S^1$. The 'interesting' part of of the Mayer-Vietoris sequence then becomes:

$$0 \to H_2(X) \xrightarrow{\delta_2} H_1(S^1 \dot\sqcup S^1) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_1)^1_* \oplus (i_2)^1_*} H_1(\mathcal{O}_1) \oplus H_1(\mathcal{O}_2) = \mathbb{Z} \oplus \mathbb{Z} \to H_1(X)$$
$$\xrightarrow{\delta_1} H_0(S^1 \dot\sqcup S^1) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_1)^0_* \oplus (i_2)^0_*)} H_0(\mathcal{O}_1) \oplus H_0(\mathcal{O}_2) = \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0.$$

Assume $\mathcal{O}_1$ is the bigger cylinder. Then the images of the two circles in the bigger cylinder bound disks and thus $(i_1)^1_* = 0$. On the other hand, each of the two circles goes to circles around the hole in the smaller cylinder which represent the same generator. Thus $(i_2)^1_*$ is surjective and the kernel is $\mathbb{Z}$. Thus $H_2(X) = \mathbb{Z}$ and we now have

$$0 \to H_1(\mathcal{O}_1) = \mathbb{Z} \to H_1(X) \xrightarrow{\delta_1} H_0(S^1 \dot\sqcup S^1) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_1)^0_* \oplus (i_2)^0_*)}$$
$$H_0(\mathcal{O}_1) \oplus H_0(\mathcal{O}_2) = \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0.$$

We now trace the arc components. The spaces $\mathcal{O}_i$ are arc connected. Thus

$$i(n_1\{a\} + n_2\{b\}) = (n_1+n_2)(\{a\} \oplus \{a\}).$$

Thus we have a sequence

$$0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$$

Since $\mathbb{Z}$ is projective, this sequence splits and we have $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$. Thus

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0,2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem #7. Prove or disprove the following assertion: "If X is the realization of a finite simplicial complex, then $\pi_2(X)$ is finitely generated."

Problem #7. This is false. Let $X = S^1 \vee S^2$. The universal cover of X is the 'German Christmas tree $\tilde{X} = \mathbb{R} \cup_n S^2$ where we join on a sphere S^2 at each integer point $n \in \mathbb{R}$. Let $C_n = [|-n|, |n|] \cup_n S^k$ and let $r : \tilde{X} \to C_n$ be the retract which the spheres $\cup_m S^2$ to $\mathbb{R}$ for $m \neq n$ and which then pushes $\mathbb{R}$ to $[-|n|, |n|]$.

Note that C_n deformation retracts to S^2 and that the isomorphism class of $\pi_2(-)$ is independent of the basepoint in an arc connected space. Let $f_n : S^2 \to C_n$ represent a non-trivial element of $\pi_k(C_n) = \pi_k(S^2)$. Then $r \circ f_m$ is the trivial element of $\pi_k(C_n, 0)$ for $n \neq m$ while $r \circ f_n$ is non-trivial. Thus, in particular, f_n does not belong to the subgroup generated by $\{f_m\}_{n \neq m}$ and thus $\pi_k(X) = \pi_k(\tilde{X})$ is not finitely generated.

Problem #8. Let X be the realization of a finite simplicial complex. Find $H_*(X;\mathbb{Z}_8)$ and $H^*(X;\mathbb{Z}_8)$ given that:

$$H_p(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0,4 \\ \mathbb{Z}_3 & \text{if } p = 1,5 \\ \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_{18} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z} & \text{if } p = 3 \end{cases}$$

Problem #8 Solution. Via the universal coefficient theorems:

$$H_p(X;\mathbb{Z}_8) = \begin{cases} \mathbb{Z}_8 & \text{if } p = 0,4 \\ 0 & \text{if } p = 1,5 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 & \text{if } p = 3 \end{cases} \oplus \begin{cases} 0 & \text{if } p = 1,5 \\ 0 & \text{if } p = 2,6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p = 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8 & \text{if } p = 4 \end{cases}$$

and

$$H^p(X;\mathbb{Z}_8) = \begin{cases} \mathbb{Z}_8 & \text{if } p = 0,4 \\ 0 & \text{if } p = 1,5 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}8 & \text{if } p = 3 \end{cases} \oplus \begin{cases} 0 & \text{if } p = 1,5 \\ 0 & \text{if } p = 2,6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p = 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8 & \text{if } p = 4 \end{cases}$$

Problem #9. Let f_1 and f_2 be disjoint embeddings of S^1 into $\mathbb{R}^3$. Determine the homology groups $H_*(\mathbb{R}^3 - f_1(S^1) - f_2(S^1))$.

Solution Problem #9. SOLUTION: By alexander duality, $\tilde{H}_*(S^3 - S^1) = \tilde{H}_*(S^1)$. Let $\mathcal{O}_i := S^3 - f_i(S^1)$. Use Mayer Vietoris to see

$$\tilde{H}_q(\mathcal{O}_1 \cap \mathcal{O}_2)) = \begin{cases} \mathbb{Z} & \text{if } q = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now take away a point to put things in S^3. Let $X = S^3 - f_1(S^1) - f_2(S^1)$, $Y = S^3 - pt$, $Z = X \cap Y$. We use Mayer Vietoris again to compute things; the $\mathbb{Z}$ in dimension 2 will disapear and we will get $H_1 = \mathbb{Z} \oplus \mathbb{Z}$, $H_0 = \mathbb{Z}$, and $H_k = 0$ for $k \geq 2$.

Problem #10. Let $\vec{a} = (a_1, a_2, ..., a_k)$ be a collection of positive integers sorted into increasing order $1 \leq a_1 \leq a_2 \leq ... \leq a_k$. Let $\mathbb{CP}(\vec{a}) := \mathbb{CP}^{a_1} \times ... \times \mathbb{CP}^{a_k}$ be a product of complex projective spaces. Show $\mathbb{CP}(\vec{a})$ is homotopy equivalent to $\mathbb{CP}(\vec{b})$ implies $\vec{a} = \vec{b}$.

Solution Problem #10. Let $\vec{a} = (a_1, a_2, ..., a_k)$ be a collection of positive integers sorted into increasing order $1 \leq a_1 \leq a_2 \leq ... \leq a_k$. Let $\mathbb{CP}(\vec{a}) := \mathbb{CP}^{a_1} \times ... \times \mathbb{CP}^{a_k}$ be a product of complex projective spaces. Show that if $\mathbb{CP}(\vec{a})$ is homotopy equivalent to $\mathbb{CP}(\vec{b})$, then $\vec{a} = \vec{b}$. ANS: The Poincare polynomial is

$$p(\vec{a}) := \prod_{1 \leq a \leq k} \frac{1 - t^{2k}}{1 - t^2}.$$

Lemma. *If $p(\vec{a}) = p(\vec{b})$, then $\vec{a} = \vec{b}$.*

Proof. Suppose the Lemma fails. If $\vec{a} = (a_1, ..., a_k)$, let the length $\ell(\vec{a}) = k$. Choose a counter example where $\max(\ell(\vec{a}), \ell(\vec{b}))$ is minimal. If $a_k = b_j$, then we could cancel one factor and get a counter example of smaller length. Thus we may suppose, say, $a_k < b_j$. Then $p(\vec{b})$ has a root which is not a root of $p(\vec{a})$. This contradiction establishes the Lemma. Applying the Lemma to the Poincare polynomial of $\mathbb{CP}(\vec{a})$ then completes the proof.