1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Winter 2002

Algebraic Topology and Differential Geometry

NAME ______________________________
(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ______________ SIGNATURE ______________

Please do any 10 problems out of the following 20.

1. Prove that a pair (X, A) of CW-complexes is a Borsuk pair.

2. Let $p : T \longrightarrow X$ be a covering space. Prove that the homomorphism $p_* : \pi_1(T, \widetilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is a monomorphism (injective).

3. Let $p : E \longrightarrow B$ be Serre fiber bundle, where B be a path connected space. Prove that the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent for any two points $x_0, x_1 \in B$.

4. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{2000} \times \mathbf{RP}^{2000} \to \mathbf{CP}^{2000} \times \mathbf{RP}^{2000}$$

be a map. Prove that f always has a fixed point.

5. Define a degree of a map $f : S^n \longrightarrow S^n$. Let $f : S^n \longrightarrow S^n$ be a map of degree $d = \deg f$. Prove that the induced homomorphism $f_* : H_n(S^n) \longrightarrow H_n(S^n)$ is the multiplication by d.

6. Show that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and that they are not homotopy equivalent.

7. Define the Eilenberg-MacLane space $K(\pi, n)$. Let π be a finite group, and $K(\pi, 1)$ be the Eilenberg-MacLane space. Compute $H^1(K(\pi, 1); \mathbf{Z})$ and $H_1(K(\pi, 1); \mathbf{Z})$ in terms of the group π.

8. State the Jordan-Brouwer Theorem. Let $S^k \subset S^n$, $0 \leq k \leq n-1$. Prove that

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases}$$

9. Let $\alpha \in \pi_{4k-1}(S^{2k})$. Define Hopf invariant $h(\alpha) \in \mathbf{Z}$. Compute the Hopf invariant of the Whitehead product $[\iota_{2k}, \iota_{2k}]$, where $\iota_{2k} \in \pi_{2k}(S^{2k})$ is a generator given by the identity map $S^{2k} \longrightarrow S^{2k}$.

10. Let M be a 2-dimensional surface, $M \neq S^2, \mathbf{RP}^2$. Prove that $\pi_q M = 0$ for all $q \geq 2$.

11. **(a)** Calculate the Frenet frame, the curvature, and the torsion for the path $\gamma(t) = (37\cos(t), 37\sin(t), t)$ in $\mathbf{R}^3$.

(b) Is there a path in $\mathbf{R}^3$ with vanishing torsion and nonvanishing curvature? If yes, give an example. If no, explain why.

(c) Is there a path in $\mathbf{R}^3$ with vanishing curvature and nonvanishing torsion? If yes, give an example. If no, explain why.

12. Determine if the surface specified as the graph of the function $f(y,z) = e^{y+z}$ on the domain $(-1,+1) \times (-1,+1)$ is umbilic (i.e. has principal curvatures equal) anywhere.

13. Determine the values of α for which the set

$$\lambda = \{(x,y)\epsilon\mathbf{R}^2 \mid y^2 = x(x-1)(x-\alpha)\}$$

is a closed, embedded submanifold of $\mathbf{R}^2$. In making your argument, carefully state any theorems you use directly.

14. Let U be an open chart on $\mathbf{R}^2$ (with coordinates (x,y)). Let $f : U \to \mathbf{R}^1$ be a smooth function, and let $g = e^{2f}(dx^2 + dy^2)$ be a Riemannian metric on U. Compute the scalar curvature corresponding to g.

15. Let M be a smooth, oriented, compact manifold with boundary.

(a) Give a careful statement of Stokes' Theorem for a smooth differential form ω on M.

(b) Let g be a Riemannian metric on M, with corresponding volume element η, star operator $\star$, and laplacian $\Delta f = \star d \star df$ for any smooth function f. Derive Green's identity

$$\int_M (f\Delta h)\eta = -\int_M g^{-1}(df, dh)\eta$$

for all smooth functions f and h which vanish on the boundary.

(c) Consider the operator Δ on the space of functions which vanish on the boundary of M. Show that all the eigenvalues of Δ are negative or zero on this space.

16. **(a)** Let $(\mathbf{R}^3, g)$, $g = dx^2 + dy^2 + dz^2$ be a 3-dimensional Euclidean space. Let $f : [-1,+1] \times [-1,+1] \times [-1,+1] \to \mathbf{R}$ be a smooth function. Derive an integral formula for the volume of the 2-dimensional surface given by the graph of f.

(b) Show every smooth manifold admits a Riemannian metric.

(c) Exhibit a smooth manifold which does not admit a Lorentzian metric.

17. Let $\gamma(s) = (x(s), y(s))$ for $s \in [0,1]$ be a smooth path in $\mathbf{R}^2$, parametrized by arclength, with $x(s) > 0$ for all s. Consider the smooth surface obtained by revolving γ around the y-axis in $\mathbf{R}^3$; it is parametrized as follows

$$\Sigma(t,s) = (\cos(t)x(s), y(s), \sin(t)x(s)).$$

Determine the values of s_0 for which the paths $\lambda(t) = \Sigma(t, s_o)$ are geodesics.

18. A manifold is homogeneous if it is diffeomorphic to G/H, where G is a Lie group and H is a closed subgroup. Show that each of the following are homogeneous by exhibiting for each a transitive group action by a Lie group G, and, for some point p in the manifold, a closed subgroup H which is the isotropy subgroup of that point p:

(a) $S^{n-1} = \{ \, x \in \mathbf{R}^n \mid |x| = 1 \, \}$

(b) $\mathbf{CP}^{n-1} = (\mathbf{C}^n - \{0\})/\sim$, where $\sim$ is the equivalence relation given by

$$(z_1, ...z_n) \sim (w_1, ...w_n)$$

if there is a nonzero complex number λ such that $z_i = \lambda w_i$ for $i = 1, ...n$.

19. Let (M, g) be a Riemannian manifold, with orthonormal frame $\{e_a\}$ and dual frame $\{E^b\}$. Let ∇ be the Levi-Civita connection for this manifold, with $\Gamma^b_{ca} = E^b(\nabla_{e_c} e_a)$, and $\omega^b_a = \Gamma^b_{ca} E^c$.

(a) Use the no torsion and metric compatibility conditions to derive $dE^c = -\omega^c_b \wedge E^b$ and $\omega_{ab} + \omega_{ba} = 0$.

(b) For $g = e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2 d\theta^2$, find a corresponding $\{E^a\}$ and $\{\omega^b_c\}$.

20. (a) Let V and W be a pair of vector fields on a manifold M. Show that $[V, W]$ defined by $[V, W]f = V(W(f)) - W(V(f))$ at each point is a vector field on M.

(b) Verify the Jacobi identity

$$[V, [W, Z]] + [W, [Z, V]] + [Z, [V, W]] = 0.$$

(c) If we define the Lie derivative as follows

$$L_V W = d/dh((\theta_{-h})_* W_{\theta_h(p)})|_{h=0}$$

where θ_h denotes the flow generated by V, show for the case $V(p) \neq 0$ that $L_V W = [V, W]$.