TOPOLOGY-GEOMETRY QUALIFYING EXAM, WINTER 2000

(1) Let X be constructed by attaching two disks to S^1, one with the attaching map of degree 2 and one with attaching map of degree 3. Show the resulting space is homotopy equivalent to S^2.

(2) Define the cross product on chains using the method of acyclic models. You may use that $\tilde{H}_*(X) = 0$ when X is contractible.

(3) If $f : X \to Y$ is an injection on H_1, need it be an injection on π_1? If $f : X \to Y$ induces a surjection on H_1, need it induce a surjection on π_1?

(4) State and prove the "path lifting property" for covering spaces.

(5) Suppose $p : X \to Y$ is a covering space, and $p(x_0) = y_0$. What is the relationship between $\pi_1(X, x_0), \pi_1(Y, y_0)$ and the group of deck transformations?

Sketch the proof of this.

(6) State the Eilenberg-Steenrod axioms for homology, and calculate the homology of S^n from the axioms.

(7) Prove that if $A \subseteq X$ is a subcomplex of a CW-complex then

$$H_*(X, A) = \tilde{H}_*(X/A).$$

(8) If X and Y have isomorphic homotopy groups, need they have isomorphic homology groups? If X and Y have isomorphic homology groups, need they have isomorphic homotopy groups?

(9) Suppose X is a simply connected CW-complex with $\tilde{H}_3(X) = \mathbf{Z}/(5)$ and $\tilde{H}_i(X) = 0$ for $i \neq 3$. Prove X is homotopy equivalent to $S^3 \cup_5 e^4$.

(10) Prove that $O(n)$ is homotopy equivalent to $GL(n)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(1) I'll call the 2-cell of X attached by $\cdot 2$ the "northern cell" and the 2-cell attached by $\cdot 3$ the "southern cell" even though X does not embed in $\mathbf{R}^3$.

Then we think of S^2 as constructed in an analogous manner with the 1-skeleton = the equator and the northern and southern hemispheres attached by $\cdot 1$.

We map S^2 to X by mapping the northern hemisphere to the northern hemisphere (consider them both as quotients of the unit disk in $\mathbf{C}$) by sending z to z^3. We map the southern hemispheres by $z \mapsto z^2$. We check that this induces $z \mapsto z^6$ on the equator (so in particular the two maps agree on the overlap).

Now note that the long exact sequence of a pair implies that $z \mapsto z^n$ induces multiplication by n on $H_2(D^2, S^1)$. The image of the generator of $C_2(S^2)$ corresponding to the northern hemisphere is the image of 1 under

$$H_2(D^2, S^1) \to H_2(S^2, S^1) \to H_2(X, X^{(1)}).$$

Here the first map is the characteristic map for the northern hemisphere of S^2. But this map factors as

$$H_2(D^2, S^1) \xrightarrow{z \mapsto z^3} H_2(D^2, S^1) \to H_2(X, X^{(1)})$$

where the second map is the characteristic map of the "northern" hemisphere of X.

So we get $(1,0) \to (3,0)$ and $(0,1) \to (0,2)$. So the generator of $H_2(S^2)$, which is $(1,-1)$ has image in $H_2(X)$ given by $(3,-2)$. But this generates $H_2(X)$. So this map is an isomorphism on H_*.

(2) We wish to define

$$\times : \Delta_p(X \times Y) \to \Delta p + q(X \times Y),$$

subject to the restriction that

$$\delta(a \times b) = (\delta a) \times b + (-1)^{|a|} a \times (\delta b).$$

We first define $\times$ when p or q is 0. If $p = 0$, and a is a singular 0 simplex whose value is $x \in X$, and b is any singular q simplex of Y, we define $a \times b$ to be the singular q simplex of $X \times Y$ defined by

$$(a \times b)(v) = (x, b(v)).$$

We make the symmetric definition if $q = 0$.

Note that this defines $\times$ for $p + q < 2$. Now assume $\times$ is defined for $p + q < n$, and let $i_j \in \Delta_j(\Delta_j)$ be the identity standard j-simplex, $i_{n-j} \in \Delta_{n-j}(\Delta_{n-j})$ the identity standard $n - j$ simplex, where $n \geq 2$ and $0 < j < n$. We wish to define

$$i_j \times i_{n-j} \in \Delta_n(\Delta_j \times \Delta_{n-j}).$$

We note that

$$\delta i_j \times i_{n-j} + (-1)^j i_j \times \delta i_{n-j} \in \Delta_{n-1}(\Delta_j \times \Delta_{n-j})$$

is already defined by induction. Furthermore by the effect of δ on $\times$, we have

$$\delta(\delta i_j \times i_{n-j} + (-1)^j i_j \times \delta i_{n-j}) = \delta^2 i_j \times i_{n-j} + (-1)^{i-1}\delta i_j \times \delta i_{n-j} + (-1)^i \delta i_j \times \delta i_{n-j} + (-1)^{2j} i_j \times \delta^2 i_{n-j}$$

which is 0 since $\delta^2 = 0$.

So since $\Delta_j \times \Delta_{n-j}$ is contractible, $H_{n-1}(\Delta_j \times \Delta_{n-j}) = 0$, and the class we just checked is a cycle must also be a boundary. So we define $i_j \times i_{n-j}$ to be some chain c so that $\delta c = \delta i_j \times i_{n-j} + (-1)^j i_j \times \delta i_{n-j}$.

Then if σ is a j simplex in X and τ is an $n-j$ simplex in Y, we define $\sigma \times \tau = (\sigma \times \tau)_\Delta(i_j \times i_{n-j})$.

This makes it easy to verify that $\times$ is natural, and that the boundary formula holds.

(3) No. As an example, consider the map $S^1 \vee S^1 \to S^1 \times S^1$.

Also no. Let Y be a space with a non-zero perfect fundamental group, and X be a point.

(4) Let $p : X \to Y$ be a covering map. Let $f : I \to Y$ be a path in Y starting at y_0. Then if x_0 is a point of $p^{-1}(y_0)$ then there is a unique path $\tilde{f} : I \to X$ such that $\tilde{f}(0) = x_0$ and $p \circ \tilde{f} = f$.

To prove this, let $\{U_\alpha\}$ be a covering of Y by evenly covered path connected sets. By the Lebesque lemma, there is a number n so that $f|_{[i/n,(i+a)/n]}$ has image in some U_α.

Suppose $\tilde{f}$ is defined on $[0, i/n]$ (the base case is $i = 0$), and suppose $f([i/n, (i+1)/n]) \subseteq U_\alpha$. Then $\tilde{f}(i/n)$ is in exactly one of the path components of $p^{-1}U_\alpha$, call that set V_α.

Let p' be p restricted to V_α, which is a homeomorphism. Then let $\tilde{f}|_{[i/n,(i+1)/n]}$ be defined by $\tilde{f}(t) = (p')^{-1}f(t)$. This is continuous since it agrees with the definition of $\tilde{f}$ at i/n, and is the unique continuous map to do that.

(5) Firstly, $p_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group monomorphism, so we can think of $\pi_1(X, x_0)$ as a subgroup of $\pi_1(Y, y_0)$.

Secondly, the subgroup is normal precisely when the group of deck transformations, Δ is isomorphic to $\pi_1(Y, y_0)/\pi_1(X, x_0)$.

To see the first fact, we apply the homotopy lifting property. If γ is a loop at x_0 in X that is contractible in Y, then we lift the contracting homotopy. This lift gives a homotopy between γ and the constant loop at x_0.

To see the fact about deck transformations, note that given x_0, x_1 in $p^{-1}(y_0)$ there is at most one map $f : X \to X$ covering the identity of Y and taking x_0 to x_1. Now if $\tilde{\alpha}$ is a path in X from x_0 to x_1, and α is $p \circ \alpha$, then we have

$$\pi_1(X, x_1) = \alpha \cup_1 (X, x_0)\alpha^{-1}$$

as subgroups of $\pi_1(Y, y_0)$.

Also, *any* $\alpha \in \pi_1(Y, y_0)$ lifts to a path in X from x_0 to *some* $x_1 \in p^{-1}y_0$. On the other hand, the map $p : X \to Y$ lifts to a homeomorphism $f : X \to X$ taking x_0 to x_1 precisely when

$$\pi_1(X, x_1) = \alpha \cup_1 (X, x_0)\alpha^{-1}.$$

So we see that we get a deck transformation taking x_0 to x_1 for any choice of x_1 precisely when $\pi_1(X, x_0)$ is normal.

Finally the isomorphism $\pi_1(Y, y_0)/\pi_1(X, x_0)$ to Δ comes from taking a path α and lifting it to $\tilde{\alpha}$ starting at x_0 and associating α to the deck transformation that takes x_0 to $\tilde{\alpha}(1)$.

(6) Axioms

dimension $H_0(*) = G$, $H_i(*) = 0$ for $i \neq 0$.

functoriality $H_*(-)$ is a functor from pairs of spaces to graded abelian groups.

exactness For any pair (X, A), there is a natural (with respect to maps between pairs of spaces) transformation $H_*(X, A) \xrightarrow{\delta} H_{*-1}(A)$ which makes the long exact homology sequence of a pair.

homotopy If $f, g : (X, A) \to (Y, B)$ are homotopic then they induce the same maps on homology.

excision If U is open with the closure of U a subset of the interior of A then the inclusion induces an isomorphism

$$H_*(X - U, A - U) \to H_*(X, A).$$

compact support (I didn't emphasize this in class) If $x \in H_p(X, A)$ there is a compact pair $(X', A') \subseteq (X, A)$ so that x is in the image of the inclusion $H_p(X', A') \subseteq H_p(X, A)$.

Using excision, we prove that $\tilde{H}_0(S^0) = G$, and the other groups are 0.

By induction assume $\tilde{H}_*(S^i) = G$ concentrated in dimension i. Let D_N^{i+1} be the part of S^{i+1} above the equator (last coordinate 0), D_S^{i+1} be the part of S^{i+1} below the equator, N and S be the north and south poles. By the 5-lemma,

$$\tilde{H}_*(S^{i+1}) = \tilde{H}_*(S^{i+1}, N) = \tilde{H}_*(S^{i+1}, D_n^{i+1}).$$

By excising an open region U, slightly smaller than the open northern hemisphere, applying excision, and then applying the homotopy axiom and 5-Lemma to the map between the LES of the pairs $(S^{i+1} - U, D_N^{i+1} - U)$ and (D_S^{i+1}, S^i) we get

$$\tilde{H}_*(S^{i+1}, D_n^{i+1}) = \tilde{H}_*(S^{i+1} - U, D_N^{i+1} - U) = \tilde{H}_*(D_S^{i+1}, S^i).$$

Finally, by using the fact that D^{i+1} is contractible, we get

$$\tilde{H}_*(D_S^{i+1}, S^i) = \tilde{H}_{*-1} S^i.$$

(7) Let $i : A \to X$ be the inclusion of the subcomplex. We compare the pair (X, A) to $(C(i), CA)$. Let U be a small neighborhood of th cone point. By excision, $(C(i) - U, CA - U) \to (C(i), CA)$ is a homology isomorphism. But $X \to C(i) - U$ and $A \to CA - U$ are both homotopy equivalences, so by the 5-lemma, $(X, A) \to (C(i) - U, CA - U)$ is a homology equivalence. So we have an isomorphism

$$H_*(X, A) \to H_*(C(i), CA).$$

Now since CA is contractible, we get an isomorphism (from the LES of the pair) $\tilde{H}_*(C(i)) \to H_*(C(i), CA)$. Finally, since CA is a contractible subcomplex of the CW complex $C(i)$ we get a homotopy equivalence

$$C(i) \to C(i)/CA \cong X/A.$$

(8) No. Examples are $\mathbf{R}P^\infty \times S^7$ and $\mathbf{R}P^7$ which have the same homotopy groups and different homology groups, and $S^1 \vee S^1 \vee S^2$ vs. $S^1 \times S^1$ which have the same homology groups but different homotopy groups.

(9) We use the Hurewicz theorem to note that $\pi_3(X) = \mathbf{Z}/5$. Let $f : S^3 \to X$ be a generator. Let p_5 be the degree 5 map from $S^3 \to S^3$. Since $f \circ p_5$ is null homotopic, we get an extension of f

$$\tilde{f} S^3 \cup_5 e^4 \to X.$$

This map clearly induces an isomorphism in homology. Since both spaces are simply connected CW complexes, the Whitehead theorem tells us that this map is a homotopy equivalence.

(10) The Gram-Schmidt process is a strong deformation retract from $GL(n)$ to $O(n)$.