1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2013

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ________________ **SIGNATURE** ________________

Please do any 10 problems out of the following 20.

1. Algebraic Topology

Problem 1.1. Let $f : S^n \to S^n$ be a map, and $\deg(f)$ be the degree of f. Prove that

$$\mathrm{Lef}(f) = 1 + (-1)^n \deg(f).$$

Problem 1.2. Define the Hopf invariant. Prove that the Hopf invariant is a homomorphism.

Problem 1.3. Let $n \geq 2$. Consider the map

$$g : S^{2n-2} \times S^3 \overset{\text{proj}}{\longrightarrow} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \overset{\text{Hopf}}{\longrightarrow} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map. Which property fails if $n = 1$?

Problem 1.4. Let $A \subset X$, and (X, A) be a CW-pair. Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \to B$ be defined as

$$p : (f : X \to Y) \mapsto (f|_A : A \to Y).$$

Prove that the map $p : E \to B$ is a Serre fiber bundle.

Problem 1.5. Let $f : \mathbf{RP}^{2n} \# \mathbf{RP}^{2n} \to \mathbf{RP}^{2n} \# \mathbf{RP}^{2n}$ be a continuous map. Prove that f always has a fixed point.

Problem 1.6. Compute the homotopy group $\pi_3(S^2 \vee S^2)$.

Problem 1.7. Let M_g^2 be a two-dimensional surface of genus $g \geq 1$ (oriented). Compute the homotopy groups $\pi_q(M_g^2)$.

Problem 1.8. State the Freudenthal Theorem. Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes of dimensions k, ℓ respectively. Let $k + \ell + 1 < p$. Prove that the simplicial complexes K and L are not linked.

Problem 1.9. Let $S^k \subset S^n$, $0 \leq k \leq n-1$. Prove that

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases}$$

Problem 1.10. Let π be an abelian group and n be a positive integer. Prove that the homotopy type of $K(\pi, n)$ is completely determined by the group π and the integer n.

2. Differential Geometry

Problem 2.1. Show that the set of orthogonal matrices of $n \times n$, defined by

$$O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : AA^T = id\}$$

is a smooth manifold, where $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices. What is the dimension of this manifold?

Problem 2.2. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $\mathbb{R}P^n$ be the set of lines in $\mathbb{R}^{n+1}$ through the origin. They are both compact smooth manifolds of dimension n.

(1) Construct a nowhere vanishing n-form on S^n to show that S^n is orientable.
(2) Use the fact above to show that $\mathbb{R}P^n$ is not orientable when n is even.

Problem 2.3. Let $M = \mathbb{C} \cup \{\infty\}$ be the extended plane of complex numbers. This is a compact, oriented smooth 2-manifold and it is diffeomorphic to the 2-sphere (you do not need to justify this statement). For any complex polynomial of degree k $(k > 0)$ with leading coefficient one, show that it defines a natural map $F : M \to M$ with degree k.

Problem 2.4. Let S be the surface $z = x^2 + y^2$ given the inherited metric from the Euclidean metric on $\mathbb{R}^3$. Show that S is geodesically complete and that the Ricci tensor is positive definite.

Problem 2.5. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^4 + x^2 + y^4 + y^2 + z^4 + z^2 = 6\}$. Show that S is a smooth 2-dimensional sub manifold of $\mathbb{R}^3$. Give S the metric inherited from $\mathbb{R}^3$ and show that S admits at least 9 distinct closed geodesics.

Problem 2.6. Let S be an m-dimensional hyper-surface in $\mathbb{R}^{m+1}$. Let $C(t)$ be a curve on S. Show that C is a geodesic if and only if $\ddot{C}$ is perpendicular to S and C is an unparametrized geodesic if and only if $\ddot{C}$ is a linear combination of the normal and of $\dot{C}$.

Problem 2.7. Prove or disprove the following assertion: "*Let ω be a smooth 1-form and let X and Y be smooth vector fields. Then*

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])\text{''}.$$

Problem 2.8. Let G be the matrix group consisting of all invertible matrices of the form

$$g = \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \quad \text{for } u, v \in \mathbb{R}, v > 0.$$

Let $\mathfrak{g}$ be the Lie algebra. Let $ad(A)B = [A, B]$ and let $\langle A, B\rangle = \mathrm{Tr}\{ad(A)ad(B)\}$ be the Killing form. Write down a basis for $\mathfrak{g}$, determine the bracket relative to this basis, and determine the Killing form.

Problem 2.9. Determine $H^*_{DeR}(SO(4))$.

Problem 2.10. Let $\mathfrak{g}$ be the Lie algebra of a connected Lie group G; identify $\mathfrak{g}$ with the left invariant vector fields on G. Let $\{e_i\}$ be a basis for $\mathfrak{g}$ and let $\{e^i\}$ be the dual basis for the space of left invariant 1-forms $\mathfrak{g}^*$. Let

$$H^p(\mathfrak{g}) := \frac{\ker(d : \Lambda^p(\mathfrak{g}^*) \to \Lambda^{p+1}(\mathfrak{g}^*))}{\mathrm{Range}(d : \Lambda^{p-1}(\mathfrak{g}^*) \to \Lambda^p(\mathfrak{g}^*))}.$$

Let $G = S^3$. Determine $H^*(\mathfrak{g}^*)$ using the definition given above.