Topology Qualifying Exam
Fall 2012

Name:

You must do the first question, which is worth 16 points, and then all but one of the others, which are worth 12 each.

1. (a) Construct a space whose fundamental group is $\mathbb{Z}/2 \ast \mathbb{Z}/3$.

Solution: Let $X = \mathbb{R}P^2$, say with its standard CW structure, and $Y$ be obtained by attaching a two-disk to $S^3$ via a degree-three map (or alternately a Lens space which is the quotient of $S^3 = \{ |z_1|^2 + |z_2|^2 = 1 \}$ by the standard diagonal action of the cube roots of unity). Take their wedge along the zero cell, yielding another CW complex. Then the standard presentation of fundamental groups from CW structures yields a fundamental group of $\langle a, b | a^2 = e, b^3 = e \rangle$, which is exactly the desired free product.

(b) Construct a cover of that space whose group of deck transformations is $S_3$, the symmetric group on three letters.

Solution: Let $\tilde{X}$ be $S^2$, which covers $\mathbb{R}P^2$ through the antipodal identification. Let $\tilde{Y}$ be the quotient of a union of three disks by identifying the boundary of the $i$th to that of the $(i + 1)$st by first applying a 120-degree rotation. (Here $i$ is to be considered modulo three). Then $\tilde{Y}$ covers $Y$ with each disk mapping to the its image under the quotient identification which defines $Y$.

Now take $Z$ to be three copies of $\tilde{X}$ and two of $\tilde{Y}$ with each copy of $\tilde{X}$ attached to both copies of $\tilde{Y}$ (say with wedge points at $(1,0,0)$ and $(-1,0,0)$) and with each copy of $\tilde{Y}$ attached to all three copies of $\tilde{X}$ (say with wedge points at the three roots of unity on the boundary).

The non-trivial deck transformation of each $\tilde{X}$ over $X$ extends to all of $Z$ over $X \vee Y$, by permuting the two $\tilde{Y}$'s and the other two copies of $\tilde{X}$, and similarly the deck transformations of each copy of $\tilde{Y}$ extend to all of $Z$ by cyclically permuting the $\tilde{X}$ while applying “the same” transformation to the other copy of $\tilde{Y}$. Thus the group of deck transformations acts transitively on $Z$, acting transitively on each $\tilde{X}$ and $\tilde{Y}$ individually while the copies of $\tilde{X}$ are permuted transitively by extensions of deck transformations of $\tilde{Y}$'s, and conversely.

Since the deck transformations act transitively, the cover is normal. The group of deck transformations must have order six (the number of preimages) and is non-abelian. In fact, if we consider the set of copies of $\tilde{X}$ we see that these are permuted in all possible ways by deck transformations named, so the group of deck transformations must be exactly $S_3$. 
(c) Does \( \mathbb{Z}/2 \ast \mathbb{Z}/3 \) have a subgroup which is free? of any rank?

**Solution:** Consider the covering space \( Z \) from the previous problem. Take a copy of the interval between each of the wedge points in each \( X \) say in an equator, and in each copy of \( \tilde{Y} \) take two intervals one from the “first” wedge point to the “second” along the boundary of the disk and another from the “second” to the “third”. Because these spaces are simply connected, the set of homotopy classes of paths between these wedge points is a singleton set, represented by paths along these intervals (or constant paths staying at a point). So the set of homotopy classes of loops in \( Z \) is the same as such homotopy classes which only travels along these intervals, which form a graph with six vertices and seven edges. So the fundamental group of \( Z \) is isomorphic to that of this graph, which is homotopy equivalent to a wedge of two circles. Because the fundamental group of a cover maps injectively under the homomorphism induced by a covering map, this implies that \( X \vee Y \) contains a subgroup which is free of rank two and thus (by a theorem proved with similar techniques) of any rank.

2. Compute the homology with integer coefficients of \( (\mathbb{R}P^7/\mathbb{R}P^4) \times \mathbb{R}P^3 \) once using cellular homology and a second time using the Künneth Theorem. (Hint: a bicomplex could help with organization).

**Solution:** For both approaches, it is helpful to use standard cell complexes of the factors. That for \( \mathbb{R}P^3 \) is (starting in degree zero, from left to right) \( \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \). For \( \mathbb{R}P^7/\mathbb{R}P^4 \) we use the fact that the standard inclusion of \( \mathbb{R}P^4 \) in \( \mathbb{R}P^7 \) is cellular in the standard cell structure. We may the use the quotient complex above degree zero (along with a \( \mathbb{Z} \) in degree zero), which the starts in degree five and proceeds as \( \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \). We compute that \( \mathbb{R}P^3 \) has homology of \( \mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z} \) in degrees 0 through 3 respectively and no homology in higher degrees. Similarly, \( \mathbb{R}P^7/\mathbb{R}P^4 \) has homology \( \mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} \) in 0 through 7 respectively and no homology in higher degrees.

For application of the Künneth theorem, we first compute the tensor product of these, which yield a \( \mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z} \) in degrees 0 through 3 from tensoring with \( H_0 \) of \( \mathbb{R}P^7/\mathbb{R}P^4 \), another \( \mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z} \) in degrees 7 through 10 from tensoring with \( H_7 \), and a \( \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2 \) in degrees 5 through 8 from tensoring with \( H_5 \). Then we also need to take Tor-terms and get a \( \mathbb{Z}/2 \) in degree 7 coming from Tor\((H_1(\mathbb{R}P^3), H_5(\mathbb{R}P^7/\mathbb{R}P^4))\).

In summary we get homology of

\[
\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2, 0, \mathbb{Z}
\]

in degrees 0 through 10 and zero in higher degrees.
To compute using cellular homology, we use the fact that the cellular chain complex of a product under the canonical CW structure is the tensor product of the cellular chain complexes, which we may consider as a bicomplex before taking the total complex. We get a copy of the cellular chain complex for $\mathbb{R}P^3$ in bi-degrees $(0, \ast)$ plus the following in bi-degrees $(5, 0)$ through $(7, 3)$

\[
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
2 & 2 & 2 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{array}
\]

By "removing 0 arrows", we see that this bicomplex splits as the direct sum of six complexes. Two $\mathbb{Z}$'s in bi-degrees $(7, 0)$ and $(7, 3)$, three $\mathbb{Z} \to \mathbb{Z}$'s starting in bi-degrees $(6, 3), (7, 2)$ and $(6, 0)$ and then one square in bi-degrees $(5, 2)$ through $(6, 3)$. The homology of these summands is immediate - $\mathbb{Z}$ in the first case, $\mathbb{Z}/2$ in the lower bi-degree in the second, and for the third we see a $\mathbb{Z}/2$ in bi-degree $(5, 1)$ and another in total degree 7 coming from the fact that the sum of generators in degrees $(5, 2)$ and $(6, 1)$ map to zero but only twice their sum is in the image of the generator in bi-degree $(6, 2)$. Collecting these summands along with the homology of $\mathbb{R}P^3$ we recover the groups found above.

3. Let $M(\mathbb{Z}/3, 1)$ be the Moore space obtained by attaching a 2-cell to $S^1$ using a degree 3 map. Determine whether the assignment which sends a space $X$ to the homology of $X \times M(\mathbb{Z}/3, 1)$ is a (generalized) homology theory. (You are free to use any formulation for the axioms of homology which yields the correct theory for finite CW-complexes, restating the question in terms of pairs or reduced theory if needed.)

**Solution:** We establish that this is a homology theory in the sense of being a homotopy functor which satisfies the Mayer-Vietoris axiom. To a map between spaces $f : X \times Y$ we take $(f \times id)_*$, where $id$ is the identity map on $M(\mathbb{Z}/3, 1)$, to be the induced map on this theory. If $f$ and $g$ are homtopic, then $f \times id$ is homtopic to $g \times id$, so we have the same induced map in this theory.
Finally, if $A$ and $B$ form an open cover of $X$, then $A \times M(\mathbb{Z}/3, 1)$ and $B \times M(\mathbb{Z}/3, 1)$ form an open cover of $X \times M(\mathbb{Z}/3, 1)$. The Mayer-Vietoris sequence for this choice of open cover yields a Mayer-Vietoris sequence for our theory.

Note: this assignment yields a generalized homology theory in any formulation, though formulating and establishing this is the case is simplest by using Mayer-Vietoris.

4. Let $M$ be a manifold, $W$ a codimension-$n$ submanifold possibly with boundary, and let $C^*_W(M; \mathbb{Z}/2)$ be mod-two cochains which are functions on chains transverse to $W$. In this setting, define $\tau_W$ to be the cochain whose value on $f : \Delta^n \rightarrow M$ is $\# f^{-1}(W)$.

Show that $\delta \tau_W = \tau_{\partial W}$.

Solution: $\delta \tau_W$ applied to some $\sigma : \Delta^{n+1} \rightarrow M$ is $\tau_W$ applied to $d\sigma$, which by definition would be the count $\#(\sigma|_{\partial \Delta^{n+1}})^{-1}(W)$.

What it means for a chain $\sigma : \Delta^{n+1} \rightarrow M$ to be transverse to $W$ is that its restriction to the interior of each face (including the entire simplex itself) is transverse to $W$. In this case, by the standard transversality theorem, $\sigma^{-1}(W)$ is a one-manifold, possibly with boundary. Boundary points come in two possible types: preimages of points in the interior of $W$ in the boundary of $\Delta^{n+1}$ and preimages of points in $\partial W$ in the interior of $\Delta^{n+1}$.

Consider each point in $\sigma|_{\partial \Delta^{n+1}}^{-1}(W)$, as counted by definition for $\delta \tau_W(\sigma)$. It must be the boundary point of some connected component of $\sigma^{-1}(W)$, which is a 1-manifold with boundary. By the classification of 1-manifolds, there is exactly one other boundary point. If that boundary point is also in $\partial \Delta^{n+1}$, the two points together contribute zero to the mod-two count of $\sigma|_{\partial \Delta^{n+1}}^{-1}(W)$. If that other point is in the interior of $\Delta^{n+1}$, then it must be a pre-image of $\partial W$ would contribute to the count defining $\tau_{\partial W}(\sigma)$. Similarly, each point in $\sigma^{-1}(\partial W)$ is paired through being the boundary of the same one-manifold with a second point which either is also in $\sigma^{-1}(\partial W)$ or is in $\sigma|_{\partial \Delta^{n+1}}^{-1}(W)$.

In the end we see that cardinals of $\sigma^{-1}(\partial W)$ and $\sigma|_{\partial \Delta^{n+1}}^{-1}(W)$ are equal modulo two to the number of connected 1-manifolds in $\sigma^{-1}(W)$ with one boundary point in each set, and thus equal to each other, establishing the desired equality.

5. (a) Show that if $f$ is a map from finite-dimensional chain complex $C_*$ over a field to itself then $\sum (-1)^n \text{tr} f = \sum (-1)^n \text{tr} f_*$.

(b) State the Lefshetz theorem for finite simplicial complexes (and simplicial maps), and briefly sketch its proof.
6. Consider the following subspaces of $\mathbb{R}^4$:

$$A = \{x, y, z, w \mid x^2 + y^2 + z^2 = 1, w = 0\}$$
$$B = \{x, y, z, w \mid x = y = z = 0\}$$
$$C = \{x, y, z, w \mid x = y = z = 1\}$$

Show that the complement of $A \cup B$ has the same cohomology groups as that of $A \cup C$, but that they have different cohomology rings. (Bonus: use differential topology to show that these complements each have fundamental group isomorphic to that of the complement of $A$ alone).

**Solution:** The complement of any collection of these spaces in $\mathbb{R}^4$ is homeomorphic to the complement of their one-point compactifications in $S^4$ as the one-point compactification of $\mathbb{R}^4$. For both $A \cup B$ and $A \cup C$, their one-point compactifications are $S^1 \sqcup S^2$, so by Alexander Duality their cohomology groups are isomorphic.

To calculate the ring structure, we first need to know what these cohomology groups are. We have $H^0 \cong \mathbb{Z}$, $H^1 \cong H_{4-1-1}(S^1 \sqcup S^2) \cong \mathbb{Z}$, $H^2 \cong H_{4-2-1}(S^1 \sqcup S^2) \cong \mathbb{Z}$ and $H^3 \cong \tilde{H}_0(S^1 \sqcup S^2) \cong \mathbb{Z}$.

Next, we find homology and cohomology representatives, using images of fundamental classes for homology and Thom classes for cohomology. Thinking of the analogue in $\mathbb{R}^3$ of the unit circle in the $xy$ plane and the $z$-axis (or translate) is highly instructive in guiding our choices.

For $H_1$ we take the sub manifold $(1 + \epsilon \cos t, 0, 0, \sin t)$, and for $H^1$ we take the submanifold of $\{x, y, z, w \mid x^2 + y^2 + z^2 < 1, w = 0\}$. These intersect at the point $(1 - \epsilon, 0, 0, 0)$ with the latter sub manifold having tangent vectors which span the first three coordinates of the tangent space and the former spanning the last. Thus, these represent homology and cohomology generators in dimension one.

For $H_2$ we take $\{x^2 + y^2 + z^2 = \epsilon\}$ while $w = 0$ when considering $B$ or $\{(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = \epsilon\}$ while $w = 0$ for $C$. For $H^2$ we take $x = y = 0$ while $z > 0$ (or respectively $x = y = 1$ while $z > 1$). These intersect at $(0, 0, \epsilon, 0)$ (respectively $(1, 1, 1+\epsilon, 0)$) and the former has tangent space which spans the first two coordinates while the latter spans the last two. Thus, these represent homology and cohomology generators in dimension two.

For $H_3$ we take all points distance epsilon away from $A$. If we intersect the cohomology representatives, when taking $B$ we get an intersection of points $(0, 0, z, 0)$ with $0 < z < 1$, checking that the $H^1$ representative spans the first three coordinates.
of the tangent space and the $H^2$ representative spans the last two. This intersects
the $H_3$ representative at the single point $(0, 0, 1 - \epsilon, 0)$, again transversally (the $H_4$
representative spanning all but the $z$ coordinate of the tangent space), establishing
that this cup product is non-zero. On the other hand, when taking $C$ the intersection
is empty, yielding a trivial cup product.

Bonus: (sketch) By transversality, any map from the one-manifold $S^1$ to $\mathbb{R}^4$ is ho-
motopic to one which avoids $B$ and $C$, and moreover any homotopy between maps
from $S^1$ is homotopic to one which avoids them.

7. For each of the following possible values, determine if there is a connected, oriented three-
manifold with those values for its first and second homology with integer coefficients.
Justify any which are by naming a three-manifold with that homology, without proof
needed. (Hint: product and connect sum constructions can come in handy).

(a) $H_1 \cong \mathbb{Z}, \ H_2 \cong \mathbb{Z}$.

**Solution:** Yes, there is such a manifold, namely $S^1 \times S^2$. The homology of
$S^n$ is $\mathbb{Z}$ in degrees 0 and $n$ and zero otherwise. Because these are free, the
Künneth theorem applies to say that homology of the product is the graded
tensor product, which is as desired.

(b) $H_1 \cong \mathbb{Z}/2, \ H_2 \cong \mathbb{Z}/2$.

**Solution:** There is no such manifold. If $H_2 \cong \mathbb{Z}/2$ (and by implicit assumption
$H_3 \cong \mathbb{Z}$) then $H^2 \cong \mathbb{Z} \oplus \mathbb{Z}/2$ by the Universal Coefficient Theorem. Poincaré
duality would imply that this is isomorphic to $H_0$, which is impossible.

(c) $H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2, \ H_2 \cong \mathbb{Z}$.

**Solution:** There is such a manifold, namely $(S^1 \times S^2) \# \mathbb{R}P^3$. By the first part
of the problem, the homology of $S^1 \times S^2$ is a copy of the integers in each degree
$\leq 3$. We may also recall that the homology of $\mathbb{R}P^3$ is a $\mathbb{Z}$ in dimensions 0 and
3 and a $\mathbb{Z}/2$ in dimension one. We claim their connect sum has the indicated
homology. One proof is to appeal to general facts about relative fundamental
classes, which can establish that the homology of the connected sum is the direct
sum in dimensions except for the top and bottom in general.

Instead, we argue directly using Mayer-Vietoris. For both $S^1 \times S^2$ and $\mathbb{R}P^3$, we
consider the standard CW structures, and when one removes a three-ball from the
middle of the three-cell the resulting space retracts onto the two-skeleton, which is $S^1 \vee S^2$ and $\mathbb{R}P^2$ respectively. The connect sum is by taking union over the boundary $S^2$'s. So we can cover the connect sum with one (open, by adding
collar neighborhoods) subspace homotopy equivalent to $S^1 \vee S^2$, and another homotopy equivalent to $\mathbb{R}P^2$, whose intersection is homotopy equivalent to $S^2$. The Mayer-Vietoris sequence in reduced homology starting with the homology of the intersection in dimension three reads

$$\cdots \rightarrow 0 \rightarrow 0 \oplus 0 \rightarrow ? \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus 0 \rightarrow ? \rightarrow 0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow ? \rightarrow 0.$$ 

Here the ?'s refer to the homology of the connect sum in degrees 3, 2 and 1 respectively. We need to calculate the homomorphism from $\mathbb{Z} \cong H_2(S^2)$ to $\mathbb{Z} \cong H^2((S^1 \times S^2) - D^3)$. But the fundamental class of boundary $S^2$ bounds the "relative fundamental class" of $(S^1 \times S^2) - D^3$. For example, if $(S^1 \times S^2) - D^3$ were triangulated then the sum of all three simplices (with appropriate signs, as guaranteed by orientability) would have boundary in the simplicial chain complex exactly given by the sum simplices in the boundary $S^2$. Alternately, this homomorphism must be zero because the resulting connect sum is an oriented three-manifold, so its third homology must be $\mathbb{Z}$.

Once we have this homomorphism is zero, we see that the second is $\mathbb{Z}$ and the first is $\mathbb{Z} \oplus \mathbb{Z}/2$ as desired.

8. Show that assigning to a CW-complex its cellular chain complex defines a functor from the category of CW-complexes and homotopy classes of cellular maps to the category of chain complexes of free abelian groups and chain homotopy classes of chain maps.

**Solution:** We take as a definition that $C^n_{CW}(X) = H_n(X^{(n)}, X^{(n-1)})$, and that the differential is the composition of $\partial$ from this group to $H_{n-1}(X^{(n-1)})$ and the map from this to the relative homology $H_{n-1}(X^{(n-1)}, X^{(n-2)})$. Because skeleta of CW complexes constitute good pairs and $X^{(n)}/X^{(n-1)}$ is a wedge of spheres, these chain groups are free.

A cellular map $f : X \rightarrow Y$ by definition sends $X^{(n)}$ to $Y^{(n)}$ and $X^{(n-1)}$ to $Y^{(n-1)}$, and so induces a homomorphism on relative homology - that is, on our chain groups. Because both the boundary homomorphism $\partial$ and the map to relative homology are natural for maps of pairs, this homomorphism commutes with boundary maps, and so defines a map of chain complexes which we denote $f_\#$.

The fact that the identity map induces the identity on cellular chain complexes follows from that fact for relative homology. Similarly, that $(f \circ g)_\# = f_\# \circ g_\#$ is an immediate consequence of the similar fact for relative homology.

Note: the problem should have been more clear that the homotopies for cellular maps are intended to be cellular maps of $X \times [0, 1]$, with its standard CW structure, to $Y$. 
With such a cellular homotopy $h$ between $f$ and $g$, by what we’ve just done that induces $h_\# : C_\ast^{CW}(X \times [0, 1]) \cong C_\ast^{CW}(X) \otimes C_\ast^{CW}([0, 1]) \rightarrow C_\ast^{CW}(Y)$. Here we are using the product CW structure on $X \times [0, 1]$, and the fact that the corresponding cellular chains are given by the graded tensor product. We use the standard CW structure on $[0, 1]$ with a one-cell we denote $o$ and two zero cells we denote $a$ and $b$, and in general we will abuse notation and use names of cells for the corresponding elements of the cellular chains.

If we define $P : C_\ast^{CW}(X) \rightarrow C_\ast^{CW}(Y)$ by $P(c) = h_\#(c \otimes o)$ then by the definition of the tensor product of chain complexes and the fact that $h_\#$ is a map of chain complexes we have

$$dP(c) = d(c \otimes o + (-1)^{|c|} c \otimes do) = dc \otimes o + (-1)^{|c|} c \otimes (a - b) = P(dc) + (-1)^{|c|}(f_\#(c) - g_\#(c)).$$

At the last step we use the fact that $h$ restricted to the zero cells at 0 and at 1 is $f$ and $g$ respectively. Thus $P$ serves as a chain homotopy between $f_\#$ and $g_\#$, showing that cellular chains indeed form a functor on (naive) homotopy categories. (Note: they also constitute a functor on more sophisticated versions of homotopy categories.)

9. Construct a two-sheeted cover of the genus-two orientable surface by the genus-three orientable surface, and compute the induced maps on cohomology. Check explicitly that these induced maps yield a ring homomorphism.

**Solution:** One approach is as follows. Consider a genus one surface with two boundary components (that is, a torus with two open disks removed). The basic facts are that if one identifies those two boundary circles in a way which reverses their induced orientations (and thus yields an orientable surface), one gets a genus-two surface, while if one takes two such surfaces and identifies each boundary circle to one in the other copy one gets a genus-three surface. To establish these rigorously one could either appeal to the classification of surfaces and some homology calculations or to some cut-and-paste diagrams. A sketch mostly suffices as well, but we will do none of these in these solutions. Using these basic facts, there is a covering map which sends each of the two genus one surface with boundary in the genus three surface to the genus two surface as the quotient.

To compute induced maps on cohomology, we use Thom intersection classes. Choose orientations which are compatible with each other under the covering map. Take $S^1$’s which form standard generators for $H_1$ of the genus two surface, call them $X$, $Y$, $Z$, $Z$ with $X$ and $Y$ intersecting in a point and $Z$ and $W$ intersecting in a (different) point, while $X$ and $Y$ are disjoint from $Z$ and $W$. (Here a standard picture is helpful.) Then from this data we have that $H^1$ of the genus two surface is generated...
by \( \tau_X, \ldots, \tau_W \) with \( \tau_X \sim \tau_Y \) and \( \tau_X \sim \tau_Y \) both generating \( H^2 \) (represented by the Thom class of any point) while all other cup products are zero (represented by the Thom class of the empty sub manifold).

We may choose cohomology representatives for the genus three surface similarly, as \( \tau_A, \ldots, \tau_F \) so that \( A, B \) and \( C, D \) and \( E, F \) each intersect each other transversely in a point and other intersections are empty. Moreover, we may choose these so that the pre images of \( X, \ldots W \) under the covering map \( f \) (which is transverse to any sub manifold), are as follows.

- \( f^{-1}X = A \cup E \)
- \( f^{-1}Y = B \cup F \)
- \( f^{-1}Z = C \cup C' \), where \( C' \) is a parallel translate of \( C \)
- \( f^{-1}W = D \).

Here a sketch is quite helpful (but we leave such as a useful exercise).

Thus \( f^*(\tau_X) = \tau_A + \tau_B \), and so forth, with the one wrinkle being that \( f^*(\tau_Z) = 2[\tau_C] \) by seeing that the orientations of \( C \) and \( C' \) agree in the sense that the tube bounding them induces the lifted orientation on one copy and the opposite orientation on the other.

Letting \( \alpha \) be a point in the genus-two surface and \( \beta \) in the genus-three, so their Thom intersection classes generate \( H^2 \) of each, we see that \( f^*[\tau_\alpha] = 2[\tau_\beta] \) because \( f \) is a two-sheeted orientation-preserving cover.

Now we can check that

\[
f^*(\tau_X \sim \tau_Y) = f^*(\tau_\alpha) = 2[\tau_\beta],
\]

while

\[
f^*(\tau_X) \sim f^*(\tau_Y) = [\tau_{A \cup E} \sim [\tau_{B \cup F}] = [\tau_{(A \cup E) \cap (B \cup F)}] = 2[\tau_\beta].
\]

Moreover,

\[
f^*(\tau_Z \sim \tau_W) = f^*(\tau_\alpha) = 2[\tau_\beta],
\]

while

\[
f^*(\tau_Z) \sim f^*(\tau_W) = [\tau_{C \cup C'} \sim [\tau_D] = [\tau_{(A \cup E) \cap D}] = 2[\tau_\beta].
\]

All other cup products of generators for the genus-two surface are zero, represented by the Thom class of the empty sub manifold ( after replacing \( A \) with a parallel \( A' \), etc), but the pre image of this intersection will then also be empty, yielding a zero result in the image as needed.