Exam

Topology Qualifying Exam
Fall 2012

Name: ______________________________

You must do the first question, which is worth 16 points, and then all but one of the others, which are worth 12 each.

1. (a) Construct a space whose fundamental group is $\mathbb{Z}/2 * \mathbb{Z}/3$.
   (b) Construct a cover of that space whose group of deck transformations is $\mathcal{S}_3$, the symmetric group on three letters.
   (c) Does $\mathbb{Z}/2 * \mathbb{Z}/3$ have a subgroup which is free? of any rank?

2. Compute the homology with integer coefficients of $(\mathbb{R}P^7/\mathbb{R}P^4) \times \mathbb{R}P^3$ once using cellular homology and a second time using the Künneth Theorem. (Hint: a bicomplex could help with organization).

3. Let $M(\mathbb{Z}/3, 1)$ be the Moore space obtained by by attaching a 2-cell to S^1 using a degree 3 map. Determine whether the assignment which sends a space X to the homology of $X \times M(\mathbb{Z}/3, 1)$ is a (generalized) homology theory. (You are free to use any formulation for the axioms of homology which yields the correct theory for finite CW-complexes, restating the question in terms of pairs or reduced theory as needed.)

4. Let M be a manifold, W a codimension-n submanifold possibly with boundary, and let $C^*_{\pitchfork W}(M; \mathbb{Z}/2)$ be mod-two cochains which are functions on chains transverse to W. In this setting, define τ_W to be the cochain whose value on $f : \Delta^n \xrightarrow{\pitchfork W} M$ is $\# f^{-1}(W)$. Show that $\delta\tau_W = \tau_{\partial W}$.

5. (a) Show that if f is a map from finite-dimensional chain complex C_* over a field to itself then $\sum(-1)^i \mathrm{tr} f = \sum(-1)^i \mathrm{tr} f_*$.
   (b) State the Lefshetz theorem for finite simplicial complexes (and simplicial maps), and briefly sketch its proof.

6. Consider the following subspaces of $\mathbb{R}^4$:

$$A = \{x, y, z, w \mid x^2 + y^2 + z^2 = 1, w = 0\}$$
$$B = \{x, y, z, w \mid x = y = z = 0\}$$
$$C = \{x, y, z, w \mid x = y = z = 1\}$$

Show that the complement of $A \cup B$ has the same cohomology groups as that of $A \cup C$, but that they have different cohomology rings. (Bonus: use differential topology to show that these complements each have fundamental group isomorphic to that of the complement of A alone).

7. For each of the following possible values, determine if there is a connected, oriented three-manifold with those values for its first and second homology with integer coefficients. Justify any which are by naming a three-manifold with that homology, without proof needed. (Hint: product and connect sum constructions can come in handy).
   (a) $H_1 \cong \mathbb{Z},\ H_2 \cong \mathbb{Z}$.
   (b) $H_1 \cong \mathbb{Z}/2,\ H_2 \cong \mathbb{Z}/2$.
   (c) $H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2,\ H_2 \cong \mathbb{Z}$.

8. Show that assigning to a CW-complex its cellular chain complex defines a functor from the category of CW-complexes and homotopy classes of cellular maps to the category of chain complexes of free abelian groups and chain homotopy classes of chain maps.

9. Construct a two-sheeted cover of the genus-two orientable surface by the genus-three orientable surface, and compute the induced maps on cohomology. Check explicitly that these induced maps yield a ring homomorphism.