1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2011

Algebraic Topology and Differential Geometry

NAME ______________________________
(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ______________ SIGNATURE ______________

Please do any 10 problems out of the following 20.

1. State and prove the Jordan-Brouwer Theorem. (If you would like to use some preliminary results, please state them clearly.)

2. Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that $h([\iota_{2n}, \iota_{2n}])$ is non-zero, and use this to prove that $\pi_{4n-1}(S^{2n})$ contains $\mathbf{Z}$.

3. Define Eilenberg-McLane space $K(\pi, n)$. Prove that $H_{n+1}(K(\pi, n); \mathrm{Z}) = 0$ for $n \geq 2$ and an arbitrary abelian group π.

4. Let $f : S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that f induces the zero map on all homotopy groups but f is not nullhomotopic.

5. Let M be a closed, orientable manifold of dimension $4k+2$. Show that the Euler characteristic of M is even.

6. Compute the homotopy groups $\pi_q(\mathrm{CP}^n)$ for $q \leq 2n+1$.

7. State the Freudenthal Theorem. Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes of dimensions k, l respectively. Let $k+l+1 < p$. Prove that the simplicial complexes K and L are not linked.

8. Define regular covering. Let X be the figure eight. Give a covering space $p : Y \to X$ and a map $f : Y \to Y$ so that $pf = p$ and f is not a homeomorphism.

9. Let $p : E \to B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

10. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathrm{CP}^{4k} \times \mathrm{RP}^{2n} \to \mathrm{CP}^{4k} \times \mathrm{RP}^{2n}$$

be a map. Prove that f always has a fixed point.

The following are differential geometry questions.

11. Let $\vec{x}(u,v) \doteqdot (u\cos v, u\sin v, f(u))$ be a smooth parametrized surface in $\mathbb{R}^3$ with $u > 0$ and $v \in (0, 2\pi)$. Compute

    (a) the first fundamental form of this surface.

    (b) the second fundamental form of this surface.

    (c) the scalar curvature of this surface using (a) and (b).

12. Find a closed differential 2-form ω on $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ such that ω is not exact. You need to justify your answer.

    If you can not do the above, you may do the following and get only half of the credits. Find a closed differential 1-form ω_1 on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$ such that ω_1 is not exact. Again you need to justify your answer.

13. Consider differential 1-form $\alpha = xdz - zdx$ on $\mathbb{R}^3$. Prove the following: if $f(x,y,z) \in C^\infty(\mathbb{R}^3)$ satisfies that $f\alpha$ is a closed 1-form, then f is identically zero. (Hint: use cylindrical coordinates.)

14. Let $M_{3\times 3}$ be the set of all 3×3 real matrices and let $SO(3) = \{A \in M_{3\times3}, AA^T = I_{3\times 3} \text{ and } \det A = 1\}$. Define the exponential map $\exp : M_{3\times3} \to M_{3\times3}$ by

$$\exp(B) = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \frac{1}{4!}B^4 + \cdots.$$

    (a) Prove that the series $\exp(B)$ converges for any 3×3 real matrix B.

    Below you may assume that exp is smooth and that $\frac{d}{ds}\exp(B(s)) = \sum_{k=1}^{\infty}\frac{1}{k!}\frac{d}{ds}(B(s))^k$ for smooth matrix-valued function $B(s)$ of $s \in \mathbb{R}$.

    (b) Show that exp is an injective map on some neighborhood of $0_{3\times3}$.

    (c) Prove that $\exp(B) \in SO(3)$ when $B \in M_{3\times3}$ satisfying $B^T = -B$. Hence we can define $Exp : \{B \in M_{3\times3}, B^T = -B\} \to SO(3)$.

    (d) Show that Exp in (c) is a surjective map from some neighborhood of $0_{3\times3}$ in $\{B \in M_{3\times3}, B^T = -B\}$ to some neighborhood of $I_{3\times3}$ in $SO(3)$. Here we do not assume $SO(3)$ has a manifold structure. (Hint: Note that every element of $SO(3)$ is a rotation around some axis, why?)

15. Let $f, g : S^1 \to \mathbb{R}^2$ be two smooth embeddings. Define set

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times \mathbb{R}^2, f(a) - g(b) = \vec{v}\}.$$

    (a) Show that M is a compact submanifold of $S^1 \times S^1 \times \mathbb{R}^2$.

    (b) Let $\pi : M \to \mathbb{R}^2$ be the projection map $\pi(a, b, \vec{v}) = \vec{v}$. Apply Sard's theorem to π to show that for almost all $\vec{v} \in \mathbb{R}^2$, $f(S^1)$ is transversal to $g(S^1) + \vec{v}$.

16. Consider the submanifold $\iota : M \to \mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 1$.

    (a) Show that the vector field

$$X = \frac{xz}{1+z^2}\frac{\partial}{\partial x} + \frac{yz}{1+z^2}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

is tangent to M.

(b) Show that the two-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ restricts to an area form on M, i.e. a two-form which never vanishes. (Hint: use cylindrical coordinates.)

(c) Does the flow of X on M preserve $\iota^*\omega$?

17. Let α be a smooth closed three-form on sphere S^6. Prove that the top form $\alpha \wedge \alpha$ vanishes at some point. (Note that the deRham cohomology $H^3_{deR}(S^6) = 0$)

18. Show that $S^1 \times S^2$ does not admit a metric of positive sectional curvature. Does your argument apply to $S^2 \times S^2$? Why? Note that one of Hopf's conjecture says that $S^2 \times S^2$ does not admit any metric of positive sectional curvature.

19. Let (M^n, g) be a complete smooth Riemannian manifold. Let $F : M \to \mathbb{R}$ be a smooth function with a lower bound. Let $C^\infty([a,b], M)$ be the set of all smooth paths from $[a,b]$ to M $(a < b)$. Define the functional $\mathcal{L} : C^\infty([a,b], M) \to \mathbb{R}$ by

$$\mathcal{L}(\gamma) = \int_a^b \left(|\gamma'|^2 + F(\gamma(t))\right) dt$$

where $\gamma' = \frac{d\gamma}{dt}$.

(a) Show that functional $\mathcal{L}$ has a lower bound.

(b) Does $\mathcal{L}$ have a upper bound? Justify your answer.

(c) Prove the following first variation formula of $\mathcal{L}$. Let γ_s, $s \in (-\epsilon, \epsilon)$, be a smooth variation of $\gamma_0 \doteqdot \gamma$.

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{L}(\gamma_s) = 2\,\langle T, S\rangle|_a^b + \int_a^b \langle -2\nabla_T T + \nabla F, S\rangle dt$$

where $T = \gamma'$ is the tangent vector field and $S = \left.\frac{d}{ds}\right|_{s=0} \gamma_s$ is the variational vector field along γ.

20. Use the moving frame method to compute the connection 1-form and the curvature 2-form of the metric $g = h(r)^2 dr^2 + f(r)^2 g_{H^{n-1}}$, where $h(r)$ and $f(r)$ are positive smooth functions and $g_{H^{n-1}}$ is the hyperbolic metric of the constant curvature -1. You may assume that $\{\bar{\omega}^i\}_{i=1}^{n-1}$ is a local orthonormal 1-form frame of $(H^{n-1}, g_{H^{n-1}})$.