Topology qualifying exam
Fall 2010
University of Oregon

1. Let $M$ be a compact, connected manifold of dimension 5. Suppose $\pi_1(M) \cong \mathbb{Z}/5$ and $H_2(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$. Compute $H^i(M)$ for all values of $i$.

2. Let $\Delta\colon S^3 \to S^3 \times S^3$ be the diagonal embedding, and $p\colon S^3 \to \mathbb{R}P^3$ the usual quotient map. Let $X$ be the space obtained by gluing $\Delta(x)$ to $p(x)$, for all $x \in S^3$:
$$X = [(S^3 \times S^3) \amalg \mathbb{R}P^3]/\sim .$$
Prove that $X$ is not homotopy equivalent to a compact manifold.

3. Suppose we have fifty-two copies of a disk $D^5$, and we glue these together in some way to make a space $X$. Assume that the gluing was done in such a way that the fifty-two maps $D^5 \to X$ are all inclusions. Prove that if $z \in H^i(X)$ and $i > 0$, then $z^{52} = 0$.

4. Let $K$ be the Klein bottle. Determine how many path-connected covering spaces of $K$ have degree 3. Pick one of these covering spaces that is non-regular and explicitly describe it by identifying the total space $E$ and the map $E \to K$.

5. Prove that $\mathbb{R}P^7$ is not homeomorphic to a product $X \times Y$ of two manifolds where $\dim X > 0$ and $\dim Y > 0$.

6. Calculate $\pi_i(\mathbb{R}P^3 \vee \mathbb{C}P^4)$ for $i \leq 2$.

7. Let $E \to B$ be a fiber bundle with fiber $F$, where $F$ is a CW-complex. For each $x \in B$, let $F_x = p^{-1}(x)$. Let $a, b \in B$ and let $\gamma\colon I \to B$ be a path from $a$ to $b$. Explain how to get a map $\gamma_\#\colon F_a \to F_b$, and prove that your construction gives a well-defined homotopy class in $[F_a, F_b]$.

8. Let $M$ be a compact, orientable $n$-manifold and let $A \subseteq M$ be any nonempty subspace. Prove that the map $H_n(M - A) \to H_n(M)$ is the zero map.

9. If $\alpha \in H^n(X)$ and $b \in H_k(X)$, define the cap product $\alpha \cap b$. Prove that if $f\colon Y \to X$ and $c \in H_*(Y)$ then $f_*(f^*(\alpha) \cap c) = \alpha \cap f_*(c)$.

10. Let $M$ be a connected 2-dimensional manifold that is also a topological group. Let $e$ be the identity, and $p\colon M \times M \times M \to M$ be given by $p(a,b,c) = abc$.

    (a) Prove that $p^*(x) = (x \otimes 1 \otimes 1) + (1 \otimes x \otimes 1) + (1 \otimes 1 \otimes x)$ for $x \in H^1(M;\mathbb{Q})$.

    (b) Prove that $\dim H^1(M;\mathbb{Q}) \leq 2$.

    [Hint for (b): Take a look at $p^*(xyz)$ for $x, y, z \in H^1(M;\mathbb{Q})$ and think about what happens if $x$, $y$, and $z$ are independent.]