1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2009

Algebraic Topology and Differential Geometry

NAME ____
(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ________________ SIGNATURE ________________

Please do any 10 problems out of the following 20.

1. Define when a pair of topological spaces (X, Y) is a Borsuk pair. Prove that a CW-pair (X, Y) is a Borsuk pair (in the case when X, Y are finite complexes).

2. Define covering space. Let $n \geq 2$. Prove that any map $f : \mathbf{RP}^n \to S^1$ is homotopic to a constant map.

3. Let $p : E \longrightarrow B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

4. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{2009} \to \mathbf{CP}^{2009}$$

be a map. Prove that f^2 has a fixed point.

5. Define the Whitehead map $w : S^{n+k-1} \to S^n \vee S^k$. Prove that the element $[w] \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

6. Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

7. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g is not homotopic to a constant map.

8. Let $X \subset S^n$ be homeomorphic to S^p, $1 \leq p \leq n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

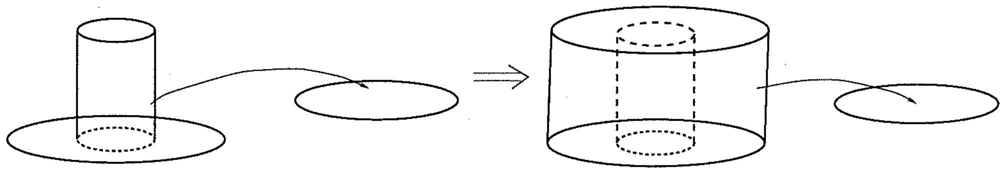
9. Let X be a finite simply-connected CW-complex with $\widetilde{H}_n(X; \mathbf{Z}) = 0$ for all n. Prove that X is contractible.

10. Let X be a path-connected space. Prove that the group $H^1(X; \mathbf{Z})$ is free abelian group.

SOLUTIONS OF problems 1-10

1. Define when a pair of topological spaces (X, Y) is a Borsuk pair. Prove that a CW-pair (X, Y) is a Borsuk pair (in the case when X, Y are finite complexes).

Solution: First, recall that a pair (of topological spaces) (X, A) a *Borsuk pair*, if for any map $F : X \longrightarrow Y$ a homotopy $f_t : A \longrightarrow Y$, $0 \leq t \leq 1$, such that $f_0 = F|_A$ may be extended up to homotopy $F_t : X \longrightarrow Y$, $0 \leq t \leq 1$, such that $F_t|_A = f_t$ and $F_0 = F$.



We are given a map $\Phi : A \times I \longrightarrow Y$ (a homotopy f_t) and a map $F : X \times \{0\} \longrightarrow Y$, such that $F|_{A\times\{0\}} = \Phi|_{A\times\{0\}}$. To extend a homotopy f_t up to homotopy F_t is the same as to construct a map $F' : X \times I \longrightarrow Y$ such that $F'|_{A\times I} = \Phi$. We construct F' by induction on dimension of cells of $X \setminus A$. The first step is to extend Φ to the space $(A \cup X^{(0)}) \times I$ as follows:

$$F'(x,t) = \begin{cases} F(x), & \text{if } x \text{ is a 0-cell from } X,\ x \notin A, \\ \Phi(x,t), & \text{if } x \in A. \end{cases}$$

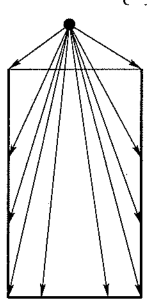
Now assume that F' is already defined on $(A \cup X^{(n)}) \times I$. Let e^{n+1} be a $(n+1)$-cell, $e^{n+1} \subset X \setminus A$. By induction, the map F' is already given on the cylinder $(\bar{e}^{n+1} = \setminus e^{n+1}) \times I$ since the boundary $\partial e^{n+1}\bar{e}^{n+1} \setminus e^{n+1} \subset X^n$. Let $g : D^{n+1} \longrightarrow X^{(n+1)}$ be a characterictic map corresponding to the cell e^{n+1}. We have to define an extension of F' from the side $g(S^n) \times I$ and the bottom base $g(D^{n+1}) \times \{0\}$ to the cylinder $g(D^{n+1}) \times I$. By definition of CW-complex, it is the same as to construct an extension of the map

$$\psi = F' \circ g : (S^n \times I) \cup (D^{n+1} \times \{0\}) \longrightarrow Y$$

to a map of the cylinder $\psi' : D^{n+1} \times I \longrightarrow Y$. Let

$$\eta : D^{n+1} \times I \longrightarrow (S^n \times I) \cup (D^{n+1} \times \{0\}$$

be a projection map of the cylinder $D^{n+1} \times I$ from a point s which is near and a bit above of the top side $D^{n+1} \times \{1\}$ of the cylinder $D^{n+1} \times I$, see the picture below:



The map η is an identical map on $(S^n \times I) \cup (D^{n+1} \times \{0\})$. We define an extension ψ' as follows:

$$\psi' : D^{n+1} \times I \xrightarrow{\eta} (S^n \times I) \cup (D^{n+1} \times \{0\} \xrightarrow{\psi} Y.$$

This procedure may be carried out independently for all $(n+1)$-cells of X, so we obtain an extension

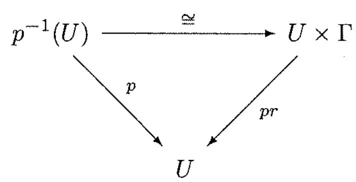
$$F' : (A \cup X^{(n+1)}) \times I \longrightarrow Y.$$

Thus, going from the skeleton $X^{(n)}$ to the skeleton $X^{(n+1)}$, we construct an extension $F' : X \times I \longrightarrow Y$ of the map $\Phi : A \times I \longrightarrow Y$.

We should emphasize that if X is an infinite-dimensional complex, then our construction consists of infinite number of steps; in that case the axiom **(W)** implies that F' is a continuous map. □

2. Define covering space. Let $n \geq 2$. Prove that any map $f : \mathbf{RP}^n \to S^1$ is homotopic to a constant map.

Solution: A path-connected space T is a *covering space* over a path-connected space X, if there is a map $p : T \longrightarrow X$ such that for any point $x \in X$ there exists a path-connected neighbourhood $U \subset X$, such that $p^{-1}(U)$ is homeomorphic to $U \times \Gamma$ (where Γ is a discrete set), futhermore the following diagram commutes



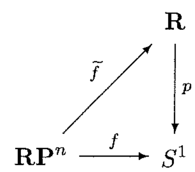
(1)

The neighbourhood U from the above definition is called *elementary neighborhood.* □

Let $f : \mathbf{RP}^n \to S^1$ be a map. We recall that $\pi_1 \mathbf{RP}^n \cong \mathbf{Z}/2$, and $\pi_1 S^1 \cong \mathbf{Z}$. Thus the induced homomorphism $f_* : \pi_1 \mathbf{RP}^n \to \pi_1 S^1$ is trivial. Consider the universal covering $\mathbf{R} \xrightarrow{p} S^1$. Let $x_0 \in S^1$ be a base point. Recall the following result:

Theorem A. *Let $p : T \longrightarrow X$ be a covering space, and Z be a path-connected space, $x_0 \in X$, $\widetilde{x}_0 \in T$, $p(\widetilde{x}_0) = x_0$. Given a map $f : (Z, z_0) \longrightarrow (X, x_0)$ there exists a lifting $\widetilde{f} : (Z, z_0) \longrightarrow (T, \widetilde{x}_0)$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0))$.*

Since the homomorphism $f_* : \pi_1 \mathbf{RP}^n \to \pi_1 S^1$ is trivial, according to Theorem A, there exists a lift $\widetilde{f} : \mathbf{RP}^n \longrightarrow \mathbf{R}$ which makes the following diagram commutative:



Now $\widetilde{f} : \mathbf{RP}^n \longrightarrow \mathbf{R}$ is null-homotopic since $\mathbf{R}$ is contractible. Thus the composition

$$f = p \circ \widetilde{f} : \mathbf{RP}^n \to S^1$$

is also null-homotopic. □

3. Let $p : E \longrightarrow B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

Solution: Let $s : I \longrightarrow B$ be a path connecting x_0 and x_1. We have to define one-to-one correspondence $\varphi_K : [K, F_0] \longrightarrow [K, F_1]$ for any CW-complex K.

Let $h_0 : K \longrightarrow F_0$ be a map. Denote $i_0 : F_0 \longrightarrow E$ the inclusion map. We have the map:

$$\widetilde{f} : K \xrightarrow{h_0} F_0 \xrightarrow{i_0} E.$$

Consider also the homotopy $F : K \times I \longrightarrow B$, where $G(x, t) = s(t)$. By the CHP there exists a covering homotopy $\widetilde{F} : K \times I \longrightarrow E$ of the map $\widetilde{f}$ such that $p \circ \widetilde{F} = F$, in particular, $\widetilde{F}(K \times \{t\}) \subset p^{-1}(s(t))$, and $\widetilde{F}(K \times \{1\}) \subset F_1$.

We define $\varphi_K(h_0 : K \longrightarrow F_0) = (h_1 : K \longrightarrow F_1)$, where $h_1 = \widetilde{F}|_{K \times \{1\}}$. We should show that the map φ_K is well-defined.

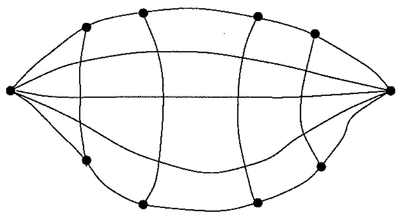
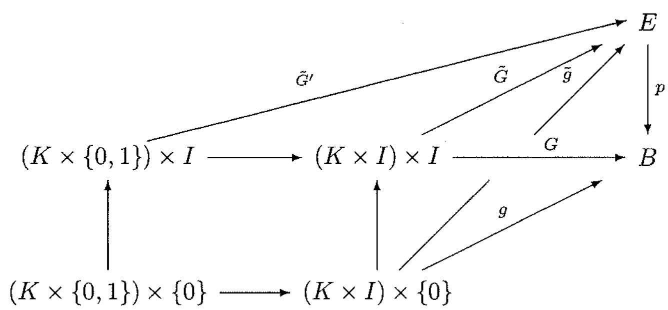


Fig. (a)

Let s' be a different path connecting x_0 and x_1, and $\widetilde{f}' : K \longrightarrow E$, $F' : K \times I \longrightarrow B$, $h' : K \longrightarrow F_1$ be corresponding maps and homotopies determined by s'. Assume that s and s' are homotopic, and let $S : I \times I \longrightarrow B$ be a corresponding homotopy. Denote by $T : I \times I \longrightarrow B$ a map defined by $T(t_1, t_2) = S(t_2, t_1)$, see Fig. (a) We are going to use the *relative version* of the CHP for the pair $Z' \subset Z$ where $Z = K \times I$ and $Z' = K \times \{0, 1\}$.

Consider the following commutative diagram:



(2)

Here the map $g : (K \times I) \times \{0\} \to B$ sends everything to x_0, and $\widetilde{g} : K \times I \longrightarrow E$ defined by $\widetilde{g}(k, t_1) = \widetilde{f}(k)$ (see above). The homotopy $G : (K \times I) \times I \longrightarrow B$ is defined by the formula: $G(k, t_1, t_2) = T(t_1, t_2)$. The map $\tilde{G}' : (K \times \{0,1\}) \times I \to E$ is defined by the homotopies F and F':

$$\tilde{G}'|_{K \times \{0\} \times I} = F, \quad \tilde{G}'|_{K \times \{1\} \times I} = F'.$$

The relative version of the CHP implies that there exists $\widetilde{G} : K \times I \longrightarrow E$ covering G and $\tilde{G}'$ as it is shown at in (2).

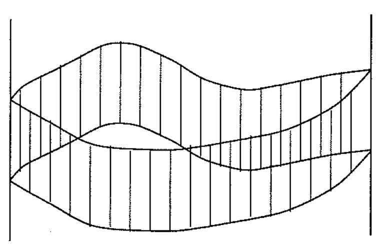


Fig. (b)

The map $(k, t) \longrightarrow \widetilde{G}(k, t, 1)$ maps $K \times I$ to F_1: this is the homotopy connecting h_1 and h'_1, see Fig. (b). Thus a path $s : I \longrightarrow B$ defines a map $\varphi_K(s) : [K, F_0] \longrightarrow [K, F_1]$, $F_0 = p^{-1}(s(0))$, $F_1 = p^{-1}(s(1))$, which does depend only of the homotopy class of s. Clearly the map φ_K is natural with respect to K; note also that if s is a constant path, then $\varphi_K = Id_{F_0}$. Moreover, if a composition of paths $s_2 \cdot s_1$ (i.e. $s_1(1) = s_2(0)$) gives a map $\varphi_K(s_2 \cdot s_1) = \varphi_K(s_2) \circ \varphi_K(s_1)$. In particular, the map $\varphi_K(s^{-1})$ is inverse to $\varphi_K(s)$: it implies that $\varphi_K(s)$ is one-to-one. $\square$

4. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbb{CP}^{2009} \to \mathbb{CP}^{2009}$$

be a map. Prove that f^2 has a fixed point.

Solution: Let A be a finitely generated abelian group. Denote $F(A)$ the free part of A, so that $A = F(A) \oplus T(A)$, where $T(A)$ is a maximum torsion subgroup of A. Let $\varphi : A \longrightarrow A$ be an endomorphism of A. We define $F(\varphi) : F(A) \longrightarrow F(A)$ by composition:

$$F(\varphi) : F(A) \xrightarrow{\text{inclusion}} A \xrightarrow{\varphi} A \xrightarrow{\text{projection}} F(A).$$

We define $\text{Tr}(\varphi) = Tr(F(\varphi))$. Now let $\mathcal{A} = \{A_q\}_{q\geq 0}$ be a *finitely generated graded abelian group*, i.e. each group A_q is finitely generated. A homomorphism $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ of two graded abelian groups is a collection of homomorphisms $\{\varphi_q : A_q \longrightarrow B_{q-k}\}$ (the number k is the *degree* of Φ).

Now let $\mathcal{A} = \{A_q\}_{q\geq 0}$ be a finitely generated graded abelian group, and let $\Phi = \{\varphi_q\} : \mathcal{A} \longrightarrow \mathcal{A}$ be an endomorphism of degree zero. We assume that $F(A_q) = 0$ for $q \geq n$ (for some n). We define the *Lefschetz number* $\text{Lef}(\Phi)$ of the endomorphism Φ by the formula:

$$\text{Lef}(\Phi) = \sum_{q\geq 0}(-1)^q \text{Tr}(\pi_q).$$

Lefschetz Fixed Point Theorem. *Let X be a finite CW-complex, $f : X \longrightarrow X$ be a map such that* $\text{Lef}(f) = 0$. *Then f has a fixed point, i.e. such point $x_0 \in X$ that $f(x_0) = x_0$.*

Now we consider a map

$$f : \mathbb{CP}^{2k-1} \to \mathbb{CP}^{2k-1}, \quad \text{where } k \geq 1.$$

Let $x \in H^2(\mathbb{CP}^{2k-1};\mathbb{Z}) \cong \mathbb{Z}$ be a generator. We know that $H^*(\mathbb{CP}^{2k-1};\mathbb{Z}) \cong \mathbb{Z}[x]/x^{2k}$. Then $f^*(x) = \lambda \cdot x$, where $\lambda \in \mathbb{Z}$. The map f induces the ring homomorphism

$$f^* : H^*(\mathbb{CP}^{2k-1};\mathbb{Z}) \to H^*(\mathbb{CP}^{2k-1};\mathbb{Z}).$$

We have: $f^*(x^q) = \lambda^q x^q$ for $q = 1, 2, \dots, 2k-1$. We notice that the Universal coefficient formula gives that the homomorphism

$$f_* : H_{2q}(\mathbb{CP}^{2k-1};\mathbb{Z}) \to H_{2q}(\mathbb{CP}^{2k-1};\mathbb{Z})$$

is also the multiplication by λ^q. Thus we have the following homomorphisms in the homology groups:

$$\begin{array}{llcccl} q=0 & H_0(\mathbb{CP}^{2k-1};\mathbb{Z}) & \cong & \mathbb{Z} \xrightarrow{Id} \mathbb{Z} & \cong & H_0(\mathbb{CP}^{2k-1};\mathbb{Z}) \text{ since } \mathbb{CP}^{2k} \text{ is connected,} \\ q=2 & H_2(\mathbb{CP}^{2k-1};\mathbb{Z}) & \cong & \mathbb{Z} \xrightarrow{\cdot\lambda} \mathbb{Z} & \cong & H_2(\mathbb{CP}^{2k-1};\mathbb{Z}) \\ \dots & \dots\dots\dots & & \dots\dots & & \\ q=4k-2 & H_{4k-2}(\mathbb{CP}^{2k-1};\mathbb{Z}) & \cong & \mathbb{Z} \xrightarrow{\cdot\lambda^{2k-1}} \mathbb{Z} & \cong & H_{4k-2}(\mathbb{CP}^{2k-1};\mathbb{Z}) \end{array}$$

Thus we have the trace

$$\text{Lef}(f) = 1 + \lambda + \lambda^2 + \dots + \lambda^{2k-1} = \frac{1-\lambda^{2k}}{1-\lambda}.$$

Then it is easy to compute the trace $\text{Lef}(f^2)$:

$$\text{Lef}(f^2) = 1 + \lambda^2 + \lambda^4 + \dots + \lambda^{4k-2} = \frac{1-\lambda^{4k}}{1-\lambda^2}.$$

Assume that $\lambda = \pm 1$. Then $\text{Lef}(f^2) = 1 + \lambda^2 + \lambda^4 + \dots + \lambda^{4k-2} \neq 0$. On the other hand, the only possibility for $\text{Lef}(f^2) = 0$ is $\lambda^{4k} = 1$ or $\lambda = \pm 1$. Thus $\text{Lef}(f^2) \neq 0$ for any map f, and the map f^2 has a fixed point. □

5. Define the Whitehead map $w : S^{n+k-1} \to S^n \vee S^k$. Prove that the element $[w] \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

Solution: Consider the product $S^n \times S^k$ as a CW-complex. We choose a cell decomposition of $S^n \times S^k$ into four cells of dimensions $0, n, k, n+k$. The first three cells give us the wedge $S^n \vee S^k \subset S^n \times S^k$.

The last cell $e^{n+k} \subset S^n \times S^k$ has the attaching map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$. This attaching map is called the *Whitehead map*. It is convenient to have a particular construction of the map w.

We can think about the sphere S^{n+k-1} as a boundary of the unit disk $D^{n+k} \subset \mathbb{R}^{n+k}$. Thus a point $x \in S^{n+k-1}$ has coordinates $(x_1, \dots, x_{n+k})$, where $x_1^2 + \cdots + x_{n+k}^2 = 1$. We define

$$\begin{aligned} U &= \left\{(x_1, \dots, x_{n+k}) \in S^{n+k-1} \mid x_1^2 + \cdots + x_n^2 \leq 1/2\right\}, \\ V &= \left\{(x_1, \dots, x_{n+k}) \in S^{n+k-1} \mid x_{n+1}^2 + \cdots + x_{n+k}^2 \leq 1/2\right\} \end{aligned}$$

The map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ is defined as follows. First we construct the maps $\varphi_U : U \longrightarrow S^n \vee S^k$ and $\varphi_V : V \longrightarrow S^n \vee S^k$ as the compositions:

$$\varphi_U : U \xrightarrow{\cong} D^n \times S^{k-1} \xrightarrow{pr} D^n \longrightarrow D^n/S^{n-1} \xrightarrow{\cong} S^n \longrightarrow S^n \vee S^k,$$

$$\varphi_V : V \xrightarrow{\cong} S^{n-1} \times D^k \xrightarrow{pr} D^k \longrightarrow D^k/S^{k-1} \xrightarrow{\cong} S^k \longrightarrow S^n \vee S^k.$$

Clearly we have that

$$\varphi_U|_{S^{n-1} \times S^{k-1}} = * = \varphi_V|_{S^{n-1} \times S^{k-1}}$$

and hence the maps φ_U, φ_V define the map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$.

Lemma A. *The Whitehed element $[w] \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism*

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

Proof. First, we state (and prove) several facts. For the exam, it is enough to state Lemmas B, C, and prove Lemma A using those results.

Lemma B. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:*

(1) $i_* : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n \times S^k)$,

(2) $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n)$,

(3) $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^k)$.

Proof of Lemma B. The exact sequence

$$\longrightarrow \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \times S^k) \longrightarrow$$

implies that $w \in \operatorname{Ker} i_*$ since $w = \partial(\iota)$.

The commutative diagram

$$\begin{array}{ccc} \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{i_*} & \pi_{n+k-1}(S^n \times S^k) \\ & \searrow^{pr_*^{(n)}} & \downarrow pr_* \\ & & \pi_{n+k-1}(S^n) \end{array}$$

(where $pr : S^n \times S^k \longrightarrow S^n$ is a map collapsing S^k to the base point) implies that $w \in \operatorname{Ker} pr_*^{(n)}$ and similarly $w \in \operatorname{Ker} pr_*^{(n)}$. ☐

Proof of Lemma A. Consider the suspension homomorphism

$$\Sigma : \pi_q(S^n \vee S^k) \longrightarrow \pi_{q+1}(\Sigma(S^n \vee S^k)).$$

Consider the commutative diagram:

$$
\begin{array}{ccccc}
\pi_{n+k-1}(S^n) & \xleftarrow{pr_*^{(n)}} & \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{pr_*^{(k)}} & \pi_{n+k-1}(S^k) \\
\downarrow \Sigma & & \downarrow \Sigma & & \downarrow \Sigma \\
\pi_{n+k}(S^{n+1}) & \xleftarrow{\Sigma(pr_*^{(n)})} & \pi_{n+k}(\Sigma(S^n \vee S^k)) & \xrightarrow{\Sigma(pr_*^{(k)})} & \pi_{n+k}(S^{k+1})
\end{array}
\tag{3}
$$

where pr denote the collapsing maps. By Lemma B $w \in \text{Ker } pr_*^{(n)}$, $w \in \text{Ker } pr_*^{(k)}$. Notice that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$. We need the following fact.

Lemma C. *There is an isomorphism*

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

Proof of Lemma C. Consider the long exact sequence for the pair $(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})$:

$$
\begin{aligned}
\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{i_*} \pi_{n+k}(S^{n+1} \times S^{k+1}) \\
\xrightarrow{j_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \longrightarrow
\end{aligned}
\tag{4}
$$

We notice that the $(n+k+1)$-skeleton of the product $S^{n+1} \times S^{k+1}$ is the wedge $S^{n+1} \vee S^{k+1}$. Thus any map $D^{k+n+1} \longrightarrow S^{n+1} \times S^{k+1}$ may be deformed to the subcomplex $S^{n+1} \vee S^{k+1}$. Thus $\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0$. The same argument gives that

$$\pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0.$$

Thus the long exact sequence (4) gives the isomorphism:

$$i_* : \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\cong} \pi_{n+k}(S^{n+1} \times S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1}). \quad \square$$

To complete the proof of Lemma A we notice that Lemma C and the diagram (3) imply that $w \in \text{Ker } \Sigma$. $\square$

6. Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

Solution: Let S^n be given as $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$. Let $n = 2k-1$, then $n+1 = 2k$. The antipodal map is given as $A : (x_1, \ldots, x_{2k}) \mapsto (-x_1, \ldots, -x_{2k})$ Let $0 \le \theta \le \pi$. Consider the homotopy

$$A_\theta = \begin{pmatrix} \cos\theta & \sin\theta & \cdots & 0 & 0 \\ -\sin\theta & \cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\theta & \sin\theta \\ 0 & 0 & \cdots & -\sin\theta & \cos\theta \end{pmatrix}$$

Then $A_0 = Id$, $A_\pi = A$. Thus $[A] = \iota_n$ if n is odd.

The case $n = 2k$ is similar. We consider the homotopy

$$A_\theta = \begin{pmatrix} \cos\theta & \sin\theta & \cdots & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\theta & \sin\theta & 0 \\ 0 & 0 & \cdots & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

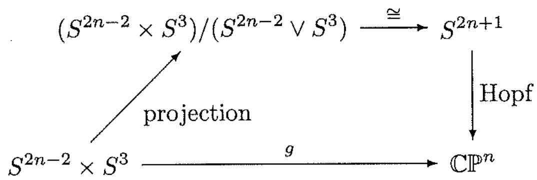
Then $A_0 = -\iota_n$, and $A_\pi = A$. $\square$

7. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{CP}^n.$$

Prove that g is not homotopic to a constant map.

Solution: We put together the map g in the following diagram:



(5)

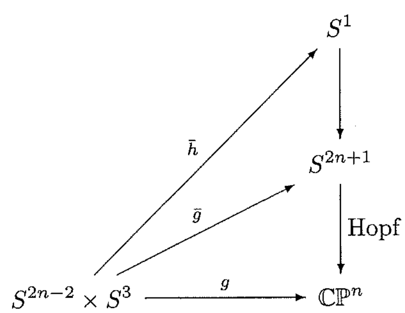
We denote by $\tilde{g}$ the composition $S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1}$. By construction, we obtain that the top $(2n+1)$-cell of the product $S^{2n-2} \times S^3$ maps to the top $(2n+1)$-cell of S^{2n+1}. In particular, it means that the induced homomorphism in homology

$$\tilde{g}_* : H_{2n+1}(S^{2n-2} \times S^3; \mathbb{Z}) \longrightarrow H_{2n+1}(S^{2n+1}; \mathbb{Z})$$

is isomorphism. Assume g is homotopic to a constant map. Then since $S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{CP}^n$ is a fiber bundle, a null-homotopy for g can be lifted to a homotopy of $\tilde{g}$ to a map $\bar{g}$,

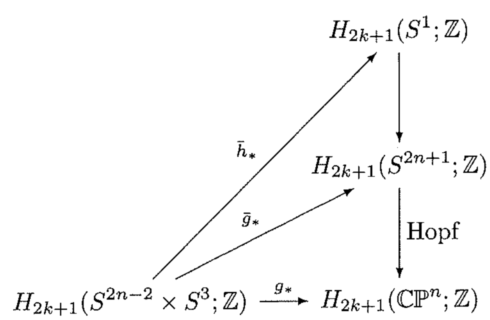
$$\bar{g} : S^{2n-2} \times S^3 \longrightarrow S^{2n+1}$$

such that the composition $\text{Hopf} \circ \bar{g}$ maps $S^{2n-2} \times S^3$ to some point $z_0 \in \mathbb{CP}^n$. It means that the map $\bar{g}$ factors through the fiber:



(6)

On the other hand, $\bar{g}_* = \tilde{g}_*$ in homology and the diagram (6) gives a commutative diagram:



Clearly $\bar{h}_* = 0$. Thus the homomorphism $\bar{g}_* = \tilde{g}_*$ must be trivial as well. Contradiction. $\square$

8. Let $X \subset S^n$ be homeomorphic to S^p, $1 \le p \le n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

Solution: First we prove a technical result.

Lemma A. *Let $K \subset S^n$ be homeomorphic to the cube I^k, $0 \le k \le n$.* Then

$$\widetilde{H}_q(S^n \setminus K) = 0 \quad \text{for all } q \ge 0.$$

Proof. Induction on k. The case $k = 0$ is obvious. Assume that the statement holds for all $0 \le k \le m-1$, and let K is homeomorphic to I^m. We choose a decomposition $K = L \times I$, where L is homeomorphic to I^{m-1}. Let $K_1 = L \times [0, \frac{1}{2}]$, and $K_2 = L \times [\frac{1}{2}, 1]$. Then $K_1 \cap K_2 = L \times \{\frac{1}{2}\} \cong I^{m-1}$. By induction,

$$\widetilde{H}_q(S^n \setminus K_1 \cap K_2) = 0 \quad \text{for all } q \ge 0.$$

We notice that the sets $S^n \setminus K_1$, $S^n \setminus K_2$ are both open in S^n. Thus we can use the Mayer-Vietoris exact sequence

$$\cdots \to \widetilde{H}_q(S^n \setminus K_1 \cup K_2) \longrightarrow \widetilde{H}_q(S^n \setminus K_1) \oplus \widetilde{H}_q(S^n \setminus K_2) \longrightarrow \widetilde{H}_q(S^n \setminus K_1 \cap K_2) \to \cdots$$

Thus we have that

$$\widetilde{H}_q(S^n \setminus K_1 \cup K_2) \cong \widetilde{H}_q(S^n \setminus K_1) \oplus \widetilde{H}_q(S^n \setminus K_2).$$

Assume that $\widetilde{H}_q(S^n \setminus K_1 \cup K_2) \ne 0$, and $z_0 \in \widetilde{H}_q(S^n \setminus K_1 \cup K_2)$, $z_0 \ne 0$. Then $z_0 = (z_0', z_0'')$, thus there exists $z_1 \ne 0$ in the group $\widetilde{H}_q(S^n \setminus K_1)$ or $\widetilde{H}_q(S^n \setminus K_2)$. Let, say, $z_1 \in \widetilde{H}_q(S^n \setminus K_1)$, $z_1 \ne 0$. Then we repeat the argument for K_1, and obtain the sequence

$$K \supset K^{(1)} \supset K^{(2)} \supset K^{(2)} \supset \cdots$$

such that

(1) $K^{(s)}$ is homeomorphic to I^m,

(2) the inclusion $i_s : S^n \setminus K \subset S^n \setminus K^{(s)}$ takes the element z to a nonzero element $z_s \in \widetilde{H}_q(S^n \setminus K^{(s)})$,

(3) the intersection $\bigcap_s K^{(s)}$ is homeomorphic to I^{m-1}.

We have that any compact subset C of $S^n \setminus \bigcap_s K^{(s)}$ lies in $S^n \setminus K^{(s)}$ for some s, we obtain that $C_q(S^n \setminus \bigcap_s K^{(s)}) = \varinjlim_s C_q(S^n \setminus K^{(s)})$, and, respectively,

$$\widetilde{H}_q(S^n \setminus \bigcap_s K^{(s)}) = \varinjlim_s \widetilde{H}_q(S^n \setminus K^{(s)}).$$

By construction, there exists an element $z_\infty \in \widetilde{H}_q(S^n \setminus \bigcap_s K^{(s)})$, $z_\infty \ne 0$. Contradiction to the inductive assumption. □

Theorem. *Let $S^k \subset S^n$, $0 \le k \le n-1$. Then*

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbb{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \ne n-k-1. \end{cases} \tag{7}$$

Proof. Induction on k. If $k = 0$, then $S^n \setminus S^0$ is homotopy equivalent to S^{n-1}. Thus the formula (7) holds for $k = 0$. Let $k \ge 1$, then $S^k = D^k_+ \cup D^k_-$, where D^k_+, D^k_- are the south and northen hemispheres of S^k. Clearly $D^k_+ \cap D^k_- = S^{k-1}$. Notice that the sets $S^n \setminus D^k_\pm$ are open in S^n, we can use the Mayer-Vietoris exact sequence:

$$\cdots \to \widetilde{H}_{q+1}(S^n \setminus D^k_+) \oplus \widetilde{H}_{q+1}(S^n \setminus D^k_-) \to \widetilde{H}_{q+1}(S^n \setminus D^k_+ \cap D^k_-) \to$$

$$\to \widetilde{H}_q(S^n \setminus S^k) \to \widetilde{H}_q(S^n \setminus D^k_+) \oplus \widetilde{H}_q(S^n \setminus D^k_-) \to \cdots$$

The groups at the ends are equal zero by Lemma A, thus

$$\widetilde{H}_q(S^n \setminus S^k) \cong \widetilde{H}_{q+1}(S^n \setminus S^{k-1})$$

since $D^k_+ \cap D^k_- = S^{k-1}$. This completes the induction. □

9. Let X be a finite simply-connected CW-complex with $\widetilde{H}_n(X;\mathbb{Z}) = 0$ for all n. Prove that X is contractible.

Solution: By assumption, $\pi_1(X, x_0) = 0$. Then the Hurewicz homomorphism $h : \pi_2(X, x_0) \longrightarrow H_2(X;\mathbb{Z})$ is an isomorphism. Thus $\pi_2(X, x_0) = 0$. Then the Hurewicz homomorphism $h : \pi_3(X, x_0) \longrightarrow H_3(X;\mathbb{Z})$ is an isomorphism, and so on. We obtain that $\pi_q(X, x_0) = 0$ for all $q = 1, 2, \ldots$. Here we use Hurewicz Theorem:

Theorem. (Hurewicz) *Let (X, x_0) be a based space, such that*

$$\pi_0(X, x_0) = 0,\ \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0, \tag{8}$$

where $n \geq 2$. Then

$$H_1(X) = 0, \quad H_2(X) = 0, \cdots, H_{n-1}(X) = 0,$$

and the Hurewicz homomorphism $h : \pi_n(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.

Thus the constant map $c : X \longrightarrow *$ induces isomorphism in homotopy groups, and Whitehead Theorem implies that X is homotopy equivalent to a point. □

10. Let X be a path-connected space. Prove that the group $H^1(X;\mathbb{Z})$ is free abelian group.

Solution: Since X is path-connected, $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, and $H^0(X;\mathbb{Z}) \cong \mathbb{Z}$. Let G be an abelian group. Then universal coefficient formula:

$$0 \longrightarrow \operatorname{Ext}(H_{q-1}(X), G) \longrightarrow H^q(X;G) \longrightarrow \operatorname{Hom}(H_q(X), G) \longrightarrow 0 \tag{9}$$

for each $q \geq 0$. If $q = 1$ and $G = \mathbb{Z}$, then $\operatorname{Ext}(H_0(X), \mathbb{Z}) = \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$, and the group

$$\operatorname{Hom}(H_1(X), \mathbb{Z})$$

does not have any torsion. Thus the group $H^1(X;\mathbb{Z}) \cong \operatorname{Hom}(H_1(X), \mathbb{Z})$ has no torsion. □