Topology Qualifying Exam
Fall 2008

Name: ____

1. (a) Determine whether the assignment which sends a space X to the graded abelian group $\mathrm{Hom}(H_*(X), \mathbb{Z}/4)$ is a cohomology theory.
   (b) Determine whether the assignment which sends a space X to the graded abelian group $H_*(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ is a homology theory.

2. Let U and V be open subsets of $\mathbb{R}^n$ and let $\Omega^\bullet(-)$ denote the de Rham complex, with $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ if $I = i_1, \ldots, i_k$. Let $F : U \times \mathbb{R} \to V$ be a smooth homotopy between $F(x, 0) = f$ and $F(x, 1) = g : U \to V$.
   (a) Suppose $F^*(\omega) = \sum \phi_I(x,t)dx_I + \sum \psi_J(x,t)dx_J \wedge dt$. Write down $f^*(\omega)$ and $g^*(\omega)$.
   (b) Define $S : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$ by
   $$\sum \phi_I(x,t)dx_I + \sum \psi_J(x,t)dx_J \wedge dt \mapsto \sum \left(\int_0^1 \psi_J(x,t)dt \right) dx_J.$$
   Show that $S \circ F^*$ serves as a chain homotopy between f^* and g^*, thus establishing the homotopy axiom for de Rham cohomology.

3. Let $M(\mathbb{Z}/6, 1)$ be the Moore space obtained by attaching a 2-disk to S^1 through a map of degree six. Endow $M(\mathbb{Z}/6, 2)$ with the structure of a Δ-complex, and use that structure to compute the cup product structure in $H^*(M(\mathbb{Z}/6,1); \mathbb{Z}/6)$ at the cochain level. You may assume knowledge of $H_*(M(\mathbb{Z}/6,1); \mathbb{Z})$.

4. State and prove the naturality property for cap product in singular homology and cohomology.

5. Let $* = [1,0,0] \in \mathbb{C}P^2$. Compute the cohomology ring of $Q = \mathbb{C}P^2 \times \mathbb{C}P^2/(x \times * \sim * \times x)$ for all $x \in \mathbb{C}P^2$. Hint: you may first want to compute the effect on homology of the quotient map from $\mathbb{C}P^2 \times \mathbb{C}P^2$ to Q.

6. Prove or disprove: the homotopy groups of a finite CW complex are finitely generated.

7. Let R be a commutative ring and let $f : M \to N$ be a map of modules over that ring. Define what it means to lift f to a map of free resolutions of M and N, and show that a lift always exists.

8. Let $M(\mathbb{Z}/n, 1)$ be the Moore space obtained by by attaching a 2-cell to S^1 using a degree n map. Construct the universal cover of $M(\mathbb{Z}/n, 1)$, showing that it is simply connected. Describe the covering map and deck transformations.

9. Let $X = S^1 \sqcup S^1$. Give two embeddings $f_1, f_2 : X \to \mathbb{R}^3$ such that $\mathbb{R}^3 - f_1(X)$ is not homeomorphic to $\mathbb{R}^3 - f_2(X)$. Justify your answer.

Topology Qualifying Exam
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Solutions

Name: ____________________

1. (a) Determine whether the assignment which sends a space X to the graded abelian group $\mathrm{Hom}(H_*(X), \mathbb{Z}/4)$ is a cohomology theory.

> **Solution:** No, it is not. Consider the sequence of spaces $S^1 \hookrightarrow \mathbb{R}P^2 \to S^2$, where $S^1 \cong \mathbb{R}P^1$ includes as the one-skeleton in the standard CW structure on $\mathbb{R}P^2$, yielding S^2 when this is collapsed. Because inclusions of skeleta of CW complexes are good inclusions, applying homology yields the long exact sequence
>
> $$\cdots H_2(\mathbb{R}P^2) = 0 \to H_2(S^2) \cong \mathbb{Z} \overset{\times 2}{\to} H_1(S^1) \cong \mathbb{Z} \to H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2 \to H_1(S^2) = 0 \cdots .$$
>
> If we apply $\mathrm{Hom}(-, \mathbb{Z}/4)$ we get the sequence
>
> $$0 \leftarrow \mathbb{Z}/4 \overset{\times 2}{\leftarrow} \mathbb{Z}/4 \leftarrow \mathbb{Z}/2 \leftarrow 0,$$
>
> which is not exact.

(b) Determine whether the assignment which sends a space X to the graded abelian group $H_*(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ is a homology theory.

> **Solution:** Yes, it is. It is the composite of two functors, namely mod-2 homology and $\otimes_{\mathbb{Z}/2}(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$, and is thus itself a functor. If f is homotopic to g then $f_* = g_*$ on mod-2 homology, and the induced maps remain the same after tensoring with $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, so this functor satisfies the homotopy axiom. Finally, since $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is a free module over $\mathbb{Z}/2$, tensoring with it preserves exactness (that is, all $\mathrm{Tor}^i_{\mathbb{Z}/2}(M, \mathbb{Z}/2 \oplus \mathbb{Z}/2)$ vanish for $i > 0$), so this functor has a long exact sequence for a pair and is thus a homology theory.

2. Let U and V be open subsets of $\mathbb{R}^n$ and let $\Omega^\bullet(-)$ denote the de Rham complex, with $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ if $I = i_1, \ldots, i_k$. Let $F : U \times \mathbb{R} \to V$ be a smooth homotopy between $F(x, 0) = f$ and $F(x, 1) = g : U \to V$.

(a) Suppose $F^*(\omega) = \sum \phi_I(x,t) dx_I + \sum \psi_J(x,t) dx_J \wedge dt$. Write down $f^*(\omega)$ and $g^*(\omega)$.

> **Solution:** f^* is the composite of F^* with the inclusion of U into $U \times \mathbb{R}$ as $U \times 0$. Thus, we set $t = 0$ and so $dt = 0$ to get that $f^*(\omega) = \sum \phi_I(x, 0) dx_I$. Similarly, for g^* set $t = 1$ and $dt = 0$ to get that $g^*(\omega) = \sum \phi_I(x, 1) dx_I$.

(b) Define $S : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$ by

$$\sum \phi_I(x,t)dx_I + \sum \psi_J(x,t)dx_J \wedge dt \mapsto \sum \left(\int_0^1 \psi_J(x,t)dt \right) dx_J.$$

Show that $S \circ F^*$ serves as a chain homotopy between f^* and g^*, thus establishing the homotopy axiom for de Rham cohomology.

Solution: We compute:

$$dS(F^*(\omega)) = \sum \left(\int_0^1 \frac{\partial \psi_J}{\partial x_i} dt \right) dx_i \wedge dx_J.$$

$$\begin{aligned} SF^*(d\omega) = SdF^*(\omega) &= S\left(\sum \frac{\partial \phi}{\partial t} dt \wedge dx_I + \frac{\partial \phi}{\partial x_j} dx_j \wedge dx_I + \frac{\partial \psi}{\partial x_i} dx_i \wedge dx_J \wedge dt \right) \\ &= \sum \left(\phi(x,1) - \phi(x,0)\right) dx_I + \left(\int_0^1 \frac{\partial \psi_J}{\partial x_i} dt \right) dx_i \wedge dx_J. \end{aligned}$$

Thus, comparing with the first part, $dSF^*(\omega) - SF^*(d\omega) = f^*(\omega) - g^*(\omega)$, as required by the definition of chain homotopy.

3. Let $M(\mathbb{Z}/6, 1)$ be the Moore space obtained by attaching a 2-disk to S^1 through a map of degree six. Endow $M(\mathbb{Z}/6, 2)$ with the structure of a Δ-complex, and use that structure to compute the cup product structure in $H^*(M(\mathbb{Z}/6,1); \mathbb{Z}/6)$ at the cochain level. You may assume knowledge of $H_*(M(\mathbb{Z}/6,1); \mathbb{Z})$.

Solution: We give $M(\mathbb{Z}/6, 1)$ a Δ-complex structure starting with a hexagon with an edge e_i, $i \in \mathbb{Z}/6$ connecting each outer vertex to the center, identifying all outer edges (each oriented "clockwise") to a single edge e. In the Δ-complex structure, the center will be "0" and then the corners will be "1" and "2" proceeding in a clockwise manner. Label the 2-simplices $A_1, \cdots, A_6$. [A picture of all this would suffice].

We may assume to know $H_*(M(\mathbb{Z}/6,1); \mathbb{Z})$ is $\mathbb{Z}$ in degree 0, $\mathbb{Z}/6$ in degree 1 and is 0 otherwise. By the Universal Coefficient Theorem, $H^*(M(\mathbb{Z}/6,1); \mathbb{Z}/6)$ is $\mathbb{Z}/6$ if $* = 0, 1, 2$ and zero otherwise, since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/6) \cong \mathbb{Z}/6$ and $\mathrm{Ext}^1(\mathbb{Z}/6, \mathbb{Z}/6) \cong \mathbb{Z}/6$.

We claim a generator of H^1 is given at the cochain level by $\gamma = e^* + e_1^* + 2e_2^* + \cdots + 5e_5^*$. Indeed, it is straightforward to check that d of this is $A_1^* + \cdots + A_6^* - A_1^* - A_2^* + 2A_2^* + \cdots = 0$. Moreover, this cochain evaluates on e, which is a cycle, with a value of 1. A generator of H^2 is any A_i^*, say A_1^* for definiteness, and any class in H^2 is determined by its value on the cycle $C = A_1 + \cdots + A_6$.

Then in calculating $\gamma \cup \gamma(A_i)$, we see that only the term $e_i^* \cup e^*$ takes on a non-zero value, since the 01 edge of A_i is e_i and its 12 edge is e. Thus $\gamma \cup \gamma(C) = 1 + 2 + \cdots + 6 = 3$ in $\mathbb{Z}/6$.

So the ring structure is $\mathbb{Z}/6[x_1, y_2]/x_1^2 = 3y_2$, with all other products zero.

4. State and prove the naturality property for cap product in singular homology and cohomology.

Solution:

Let $f : X \to Y$, $x \in H_n(X)$ and $\alpha \in H^k(Y)$. The naturality property for cap products is that $f_*(f^*\alpha \cap x) = \alpha \cap f_*(x)$. Because cap product at the chain/cochain level is linear and passes to homology and cohomology, it suffices to establish that for σ a generator of $C_n(X)$ (that is, by abuse we say $\sigma : \Delta^n = [v_0, \ldots, v_n] \to X$ and $\psi \in C^k(Y)$, we have $f_*(f^*\psi \cap \sigma) = \psi \cap f_*(\sigma)$.

The proof is a matter of unraveling definitions. The right-hand side is

$$\psi\left(f \circ \sigma|_{[v_0,\ldots,v_k]}\right) f \circ \sigma|_{[v_k,\ldots,v_n]}.$$

The left-hand side is

$$f_*\left(f^*\psi(\sigma|_{[v_0,\ldots,v_k]})\sigma|_{[v_k,\ldots,v_n]}\right),$$

which by linearity and definition of f_* equals

$$\left(f^*\psi(\sigma|_{[v_0,\ldots,v_k]})\right) f \circ \sigma|_{[v_k,\ldots,v_n]},$$

which by definition of f^* is equal to our expression for the right-hand side.

5. Let $* = [1, 0, 0] \in \mathbb{C}P^2$. Compute the cohomology ring of $Q = \mathbb{C}P^2 \times \mathbb{C}P^2 / (x \times * \sim * \times x)$ for all $x \in \mathbb{C}P^2$. Hint: you may first want to compute the effect on homology of the quotient map from $\mathbb{C}P^2 \times \mathbb{C}P^2$ to Q.

Solution: Let e_0, e_2, and e_4 denote the cells of the first factor of $\mathbb{C}P^2$ in its standard CW structure, and let e_0', e_2', and e_4' denote the cells of the second factor. Then the cells of Q are given by $e_0 \times e_0'$ (dim 0), $e_0 \times e'^2 \sim e_2 \times e_0'$ (dim 2), $e_0 \times e_4' \sim e_4 \times e_0'$ and $e_2 \times e_2'$ (dim 4), $e_2 \times e_4' \sim e_4 \times e_2'$ (dim 6), and $e_4 \times e_4'$ (dim 8). The quotient map from $\mathbb{C}P^2 \times \mathbb{C}P^2$ to Q is the obvious one given by imposing the identifications $\sim$. Since the cellular chain complexes of Q and $\mathbb{C}P^2 \times \mathbb{C}P^2$ are isomorphic to their homology, those groups are free. We immediately see that the quotient map is an isomorphism on H_0, is represented by the matrix $\begin{bmatrix} 1 & 1 \end{bmatrix}$ on H_2, is represented by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ on H_4, by $\begin{bmatrix} 1 & 1 \end{bmatrix}$ on H_6, and is an isomorphism on H_8.

Because the homology groups are free, the universal coefficient theorem says that $H^*(-) \cong \text{Hom}(H_*(-), \mathbb{Z})$ in both cases. Moreover, we may compute the induced maps on cohomology as the linear duals of the induced maps on homology. Since the induced maps on homology are surjective, those on cohomology will be injective, allowing us to understand $H^*(Q)$ as a subring of $H^*(\mathbb{C}P^2 \times \mathbb{C}P^2) \cong \mathbb{Z}[x_2, y_2]/(x^3 = y^3 = 0)$ by the multiplicative version of the Künneth theorem (using the freeness of the cohomology groups). By our computation of the induced map on homology, this subring is linearly spanned by

$$1, x + y, x^2 + y^2, xy, x^2y + xy^2, x^2y^2.$$

(Indeed, we are seeing only the symmetric polynomials in x and y.) Abstractly, this ring is given by $\mathbb{Z}[a_2, b_4, c_4, d_6, e_8]$ modulo the relations $a^2 = b + 2c$, $ab = ac = d$, $ad = 2e$, $bc = 0$.

6. Prove or disprove: the homotopy groups of a finite CW complex are finitely generated.

Solution:

This is false. Consider $X = S^1 \vee S^2$, which can be given a CW complex structure with only three cells. We use the fact that for $i > 1$, $\pi_i(X) \cong \pi_i(\tilde{X})$, where $\tilde{X}$ is the universal cover of X.

We claim that the universal cover of $S^1 \vee S^2$ is $\mathbb{R} \cup (S^2 \times \mathbb{Z})/\sim$ where for each integer n, the point $n \in \mathbb{R}$ is identified with the south pole of $S^2 \times n$. The covering map sends $\mathbb{R}$ to S^1 through the standard covering map and sends each copy of S^2 to S^2 by the identity map. That $\tilde{X}$ is simply connected follows from repeated application of the van Kampen theorem to get that the subspace which is the image of $[-n, n] \cup (S^2 \times [-n, n] \subset \mathbb{Z})/\sim$ is simply connected. Then note that any map from S^1 into $\tilde{X}$ has image in one of these, to deduce that $\tilde{X}$ itself is simply connected.

Finally, we claim that $\pi_2(\tilde{X})$ is not finitely generated. Indeed, we put the obvious cell structure on $\tilde{X}$ where each 2-cell has the zero cell at some integer point in $\mathbb{R}$ as its boundary and thus has no one-cells in its boundary. From this we compute that $H_2 \cong \mathbb{Z}^\infty$. We then cite the Hurewicz theorem to see that this is also π_2.

7. Let R be a commutative ring and let $f : M \to N$ be a map of modules over that ring. Define what it means to lift f to a map of free resolutions of M and N, and show that a lift always exists.

Solution: The definition depends on your definition of resolution. One possibility is that F_*^M is defined to be a chain complex equipped with a map $\eta_M : F_*^M \to M(0)$ which induces an isomorphism on homology, where $M(0)$ is the chain complex with M in degree zero and which is zero otherwise. In this case, a lift is just a map of chain complexes $\tilde{f}$ which makes the following diagram commute.

$$\begin{array}{ccc} F_*^M & \xrightarrow{\tilde{f}} & F_*^N \\ {\scriptstyle \eta_M}\downarrow & & {\scriptstyle \eta_N}\downarrow \\ M(0) & \xrightarrow{f} & N(0). \end{array}$$

One may equivalently say that a lift is a map of augmented chain complexes which in degree -1 is just $f : M \to N$ and which induces f in homology in degree zero when restricted to unaugmented chain complexes.

To show that a lift exists we use the fact that if free modules are projective. That is, if F is free, $g : F \to B$ is a module homomorphism and $p : A \to B$ is surjective then there is a map $\tilde{g} : F \to A$ such that $g = p \circ \tilde{g}$. If we apply this lemma where $F = F_0^M$ (which is free), $g = f \circ \eta_M$ and $p = \eta_N$ (which is surjective) we get a map $\tilde{f}_0 = \widetilde{f \circ \eta_M} : F_0^M \to F_0^N$. Because the square

$$\begin{array}{ccc} F_0^M & \xrightarrow{\tilde{f}_0} & F_0^N \\ \downarrow{\scriptstyle \eta_M} & & {\scriptstyle \eta_N}\downarrow \\ M & \xrightarrow{f} & N \end{array}$$

V commutes, and η_M and η_N are surjective we see that $\tilde{f}_0$ induces f on homology.

Inductively, assume $\tilde{f}_{n-1}$ has been constructed. To construct $\tilde{f}_n$ we apply the lifting lemma with $F = F_n^M$ (which is free by definition), and $g = \tilde{f}_{n-1} \circ d$, $B = \ker d : F_{n-1}^N \to F_{n-2}^N$, and $p = d : F_n^N \to F_{n-1}^N$ (which is surjective because $F_\bullet^N$ has no homology in positive degrees.) By construction, $\tilde{f}_n : F_n^M \to F_n^N$ makes the square

$$\begin{array}{ccc} F_n^M & \xrightarrow{\tilde{f}_n} & F_n^N \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\ F_{n-1}^M & \xrightarrow{\tilde{f}_{n-1}} & F_{n-1}^N \end{array}$$

commute.

So the $\tilde{f}_\bullet$ form a map of chain complexes, which must induce the zero map on homology in positive degrees. By construction, $\tilde{f}_0$ induces f on H_0, so we have a lift.

8. Let $M(\mathbb{Z}/n, 1)$ be the Moore space obtained by by attaching a 2-cell to S^1 using a degree n map. Construct the universal cover of $M(\mathbb{Z}/n, 1)$, showing that it is simply connected. Describe the covering map and deck transformations.

Solution: Consider D^2 as the unit disk in $\mathbb{C}$. Let Z_n be obtained by taking n copies of D^2 and then identify $x \in \partial D^2_{(j)}$ with $x \in \partial D^2_{(j)}$ for all i and j. This space is simply connected by an inductive van Kampen argument: assuming Z_{n-1} is simply connected (it clearly is for $n = 1$ since D^2 is contractible), we have Z_n is the union of Z_{n-1} and D^2 over a connected space, so it is as well. The covering map down to $M(\mathbb{Z}/n, 1)$ is the composite sending $D^2_{(j)}$ first to D^2 by rotation by $2\pi j/n$ followed by the quotient map defining the Moore space. The deck transformations are generated by one which sends each $D^2_{(i)}$ to $D^2_{(i+1)}$ and then rotates every disk by $2\pi/n$. That these are all deck transformations follows from the fact that they form a group isomorphic to $\mathbb{Z}/n$ and act transitively on this n-sheeted cover.

9. Let $X = S^1 \sqcup S^1$. Give two embeddings $f_1, f_2 : X \to \mathbb{R}^3$ such that $\mathbb{R}^3 - f_1(X)$ is not homeomorphic to $\mathbb{R}^3 - f_2(X)$. Justify your answer.

Solution: Two embeddings which link the two copies of S^1 in "different" ways will generally not have homeomorphic complements. For example, consider the "unlinked" embedding given by $\{(x-2)^2 + z^2 = 1; y = 0\} \bigcup \{(x+2)^2 + z^2 = 1; y = 0\}$. Next consider the "linked" embedding which on all but a small neighborhood of the wedge point agrees with the circles $\{x^2 + y^2 = 1; z = 0\} \bigcup \{(x-1)^2 + z^2 = 1; y = 0\}$. [Note: it would suffice to draw embeddings].

We show the complements, K_1 and K_2 respectively, of these embeddings are not homeomorphic by showing they are not homotopy equivalent, through computing their cohomology rings. [Note: one could also use the fundamental group.] By Alexander duality, the ith cohomology group of the complement of any embedding of $S^1 \sqcup S^1$ in $\mathbb{R}^3$ is isomorphic to the $2-i$th reduced homology group of $S^1 \sqcup S^1 \sqcup \text{pt.}$, so we have $H^0(K_i) \cong \mathbb{Z}$, $H^1(K_i) \cong \mathbb{Z}^2$, $H^2(K_i) \cong \mathbb{Z}^2$.

To compute the ring structure of the complements, which are open subsets of $\mathbb{R}^3$ and thus manifolds, we make extensive use of intersection theory, switching to real coefficients where the theorems were established in class (though all the statements below are true with integer coefficients as well).

In the first case, consider the disks $D_1 = \{(x-2)^2 + z^2 < 1; y = 0\}$ and $D_2 = \{(x+2)^2 + z^2 < 1; y = 0$ along with the circles $C_1 = \{y^2 + (z-1)^2 = \epsilon; x = 2\}$ and $C_2 = \{y^2 + (z-1)^2 = \epsilon; x = -2\}$ [Again, these can just be drawn]. The Thom classes of the two-dimensional disks, which are proper submanifolds, define classes in $H^1(X_1)$. The fundamental classes of the small circles, which are closed submanifolds, define classes in $H_1(X_1)$. A Thom class evaluated on a fundamental class is equal to the number of points in the intersection of the manifolds which define them, if that intersection is transversal. In this case intersections are transversal, since the tangent vectors can be taken to be parallel to the coordinate axes at the intersection points [transversality does not need to be justified for full credit]. We see that $\#D_i \cap C_j = 1$ if $i = j$ but is 0 if $i \neq j$. Thus these homology classes and cohomology classes are linearly independent, and thus each form a basis.

The wedge product of Thom classes is a Thom class for their intersection, if that intersection is transversal. Because D_1 and D_2 do not intersect, the product of their Thom classes is zero. Because they are classes in H^1, their products with themselves is also zero. Thus, there are no non-trivial products in the cohomology of K_1.

For K_2, its first homology and cohomology groups have a similar basis of fundamental classes of longitudinal circles and Thom classes of disks which "fill in" the removed circles: $D_1 = \{x^2 + y^2 < 1; z = 0\}$, $D_2 = \{(x-1)^2 + z^2 < 1; y = 0\}$. But these two disks do intersect (transversally), namely in the line segment from $(0,0,0)$ to $(1,0,0)$. We claim the Thom class of this line segment defines a non-trivial element of H^2. Again by intersection theory it suffices to find a closed, oriented submanifold which intersects this segment transversally in a single point, which a torus "enclosing" one of the removed circles does. Thus, there is a non-trivial cup product in $H^*(K_2)$, meaning that K_1 and K_2 cannot be homeomorphic.