1. Computational problems (differential geometry):

Problem 1.1. Let M be a compact orientable differentiable manifold without boundary. Show that M is not contractible to a point.

Problem 1.2. Let H^2 be the upper half plane, that is $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Consider in H^2 the following inner product. If $(x,y) \in \mathbb{H}^2$ and if $u, v \in T_p\mathbb{H}^2$, then $\langle u, v\rangle_p = (u \cdot v)/y^2$ where $u \cdot v$ is the canonical inner product on $\mathbb{R}^2$. Prove that the Gaussian curvature of $\mathbb{H}^2$ has $K \equiv -1$.

Problem 1.3.

(1) Show that the equation $x^6+x^2+y^6+y^2+z^6+z^2 = 6$ determines a smooth submanifold of $\Sigma \subset \mathbb{R}^3$.
(2) Show that Σ has at least 9 distinct non-trivial closed geodesics.

Problem 1.4. There is a 2-dimensional Lie group which has a left invariant metric and an orthonormal basis $\{e_1^L, e_2^L\}$ for the Lie-algebra of left invariant vector fields so that $[e_1^L, e_2^L] = e_2^L$. Compute the scalar curvature K.

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2. Theoretical problems (differential geometry):

Problem 2.1.

(1) Give a careful statement of the Hopf-Rinow theorem – this result concerns completeness.
(2) Prove or disprove the following assertion: "If any two points of a Riemannian manifold can be joined by a unit speed geodesic σ with $\sigma(0) = P$ and $\sigma(d) = Q$ where d is the geodesic distance between P and Q, then M is a complete metric space.

Problem 2.2.

(1) Give a careful definition of the Ricci tensor $\rho(x,y)$.
(2) Give a careful statement of Meyer's theorem.
(3) Show that $\mathbb{R}^2$ does not admit a complete Riemannian metric of scalar curvature $\tau \geq 1$.

Problem 2.3. Let ϕ_t be the flow of a vector field X on a connected Riemannian manifold M. At least one of the following assertions is true and at least one is false. Prove the true assertion(s) and give counter examples for the false assertion(s). You should not attempt to both prove an assertion and provide a counter example to an assertion.

(1) If X has constant length and if the integral curves ϕ_t are geodesics, then the flows ϕ_t are isometries.

(2) If the integral curves of X are geodesics and if the flows ϕ_t are isometries, then X has constant length.
(3) If X has constant length and if the flows ϕ_t are isometries, then the integral curves of X are geodesics.

Problem 2.4.

(1) Define the DeRham cohomology groups – this relates to the exterior derivative d.
(2) Give a careful statement of the Hodge decomposition theorem – this involves the eigenspaces of the Laplacian on p-forms.
(3) Give a careful statement of the Hodge-DeRham theorem – this relates to harmonic p-forms.
(4) State the homotopy axiom (property) for DeRham cohomology.
(5) Let G be a compact connected Lie group which is equipped with a biinvariant metric. Let $g \in G$. Show that if ϕ is a harmonic p-form, then $L_g^*\phi = \phi$.

Problem 2.5.

(1) Let (M, g) be a compact connected Riemannian manifold. Let ϕ be a smooth isometry of M. Suppose there is a point P of M so $\phi(P) = P$ and $d\phi(P) = id$. Show $\phi = id$.
(2) Let G be a compact connectedt Lie group. Show G admits a bi-invariant Riemannian metric.

Problem 2.6. Let M be a Riemannian manifold.

(1) Give a careful definition of the notion of a Jacobi vector field along a geodesic σ.
(2) Give a careful definition of what it means for P and Q to be conjugate along a geodesic σ.
(3) Let σ be a geodesic from P to Q. Let $I_\sigma(X, Y) = \int g(\dot{X}, \dot{Y}) - R(X, \dot{\sigma}, \dot{\sigma}, Y)$ where X and Y are piecewise smooth perpendicular vector fields along σ vanishing at the endpoints; this is the index form along σ. State a theorem which relates properties of the index form to the existence or non-existence of conjugate points along σ.
(4) Show that if P is conjugate to a point R in the interior of σ, then Q is conjugate to a point S in the interior of σ