Topology-Geometry Qual, Fall 2007

(1) (a) Define what it is to be a *good pair*.
(b) Prove that if A is a sub cell complex of X then (X, A) is a good pair.
(c) Prove that if (X, A) are as above, then $\tilde{H}_*(X/A) \cong H_*(X, A)$. You may assume the long exact sequence of a pair in homology.

(2) (a) Let X be path connected, and locally simply connected and path connected with basepoint x_0. State the classification theorem for isomorphism classes of covering spaces $(\tilde{X}, x_1)$ of (X, x_0).
(b) Consider the subgroup H of $F\langle a, b\rangle$ generated by

$$\{a, b^2, b^{-1}a^2b, b^{-1}a^{-1}bab\}.$$

Prove or disprove: H is normal.

(3) Give a Δ-complex structure on K, the Klein bottle. Use this to compute $H^*(K; R)$ for $R = \mathbf{Z}$ and $\mathbf{Z}/(2)$ including the cup product structure.

(4) Prove that

$$S^2 \times S^2 \not\simeq S^2 \vee S^2 \vee S^4$$

but that

$$\Sigma(S^2 \times S^2) \simeq \Sigma(S^2 \vee S^2 \vee S^4)$$

(5) Let $p : \tilde{X} \to X$ be a covering map between path connected spaces. If $f : \tilde{X} \to \tilde{X}$ is a continuous map with $pf = p$, must f be a homeomorphism? Either prove, or provide a counterexample.

(6) If X is a finite cell complex, is $\pi_2(X)$ a finitely generated abelian group? Prove, or provide a counterexample.

(7) Let $f : X \to X \times I$ be $f(x) = (x, 0)$, $g : X \to X \times I$ be $g(x) = (x, 1)$. Consider

$$f_*, g_* : C_*(X) \to C_*(X \times I)$$

and prove using acyclic models that these two maps are chain homotopic. (You may assume $H_*(X) = 0$ if X is contractible.)

(8) Compute $H^*((\mathbf{R}P^7/\mathbf{R}P^4) \times \mathbf{R}P^4)$.

(9) Prove there is no retraction $\mathbf{R}P^n \to \mathbf{R}P^k$ for $k < n$.

(10) Prove or provide a counterexample: If $f : X \to Y$, then if $f_* : \tilde{H}_*(X) \to \tilde{H}_*(Y)$ is the zero homomorphism, so is $f^* : \tilde{H}^*(Y) \to \tilde{H}^*(X)$.