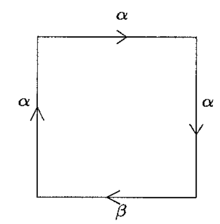
Topology Qualifying Exam
Fall 2006

This is a three-hour exam. There are nine problems, all weighted equally. When writing the solutions, you do not have to copy the problem.

1. Let $X = (I \times I)/\sim$ be the quotient space depicted by the following diagram. Prove that X has exactly one connected 2-fold covering space (up to isomorphism).



2. Let $X = S^n/\sim$ be the space obtained from S^n by identifying the north and south pole. Compute the groups $H_n(X, X-p)$ for every point $p \in X$.

3. Let X be the space from problem #2. Compute $\pi_1(X \vee \mathbb{R}P^4)$, where the basepoint in X is the point where the north and south poles were identified.

4. Let T be the torus, $p \in T$ any point, and let $N \subseteq T$ be a small open ball around p. Let $f\colon S^2 \to S^2$ be a degree 2 map, and define a space

$$X = [(T \times D^3) \amalg (T \times S^2)]/\sim$$

by gluing the point $(n, x) \in N \times S^2 \subseteq T \times D^3$ to the point $(n, f(x)) \in T \times S^2$. Compute all the homology groups of X.

5. Let X be a topological space and let $x \in X$.
   (a) Give a complete definition of the set $\pi_n(X, x)$ for $n \geq 1$, and also define the group operation (you do not need to prove anything, just give the definition).
   (b) Let $p\colon E \to B$ be a fiber bundle, $b \in B$, $F = p^{-1}(b)$, and $e \in F$. Prove that for $k \geq 1$ the induced maps

$$\pi_k(F, e) \xrightarrow{i_*} \pi_k(E, e) \xrightarrow{p_*} \pi_k(B, b)$$

   give a sequence which is exact at the middle spot. (Here $i\colon F \hookrightarrow E$ is the inclusion).

6. Using cohomology (or otherwise), prove that if $f\colon \mathbb{C}P^4 \to \mathbb{R}P^5 \times S^3$ is any map then the induced map $f_*\colon H_8(\mathbb{C}P^4) \to H_8(\mathbb{R}P^5 \times S^3)$ is zero.

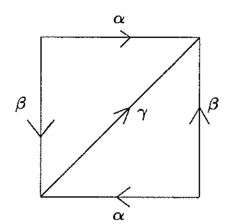
7. Let X be a space, and let $\alpha \in C^p(X)$, $\beta \in C^q(X)$, and $z \in C_n(X)$ (these groups denote singular cochains and chains, respectively).

   (a) Define $\alpha \cup \beta$ and $z \cap \alpha$.

   (b) Prove that if $f\colon X \to Y$ is a map and $\gamma \in C^q(Y)$ then $f_*(z \cap f^*\gamma) = f_*(z) \cap \gamma$.

8. Describe (perhaps by drawing a picture) the universal covering space of $\mathbb{R}P^2 \vee S^1$. Also, prove that $\mathbb{R}P^2 \vee S^1$ does not have a regular, connected covering space whose automorphism group (i.e., the group of Deck transformations) is $(\mathbb{Z}/2)^3$.

9. Consider the following Δ-complex M (a quotient space of $I \times I$):



Compute the mod 2 cohomology groups, and write down explicit cocycles representing their generators. Compute all the cup products of your generators (you don't have to show your work for the obvious ones). It turns out that M is a 2-manifold, and so it has a mod 2 fundamental class. Compute the cap products of all your generators with the fundamental class.