1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2005

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ SIGNATURE ____________________

Please do any 10 problems out of the following 20.

1. Define when a pair of topological spaces (X, Y) is a Borsuk pair. Prove that a CW-pair (X, Y) is a Borsuk pair (in the case when X, Y are finite complexes).

2. Define covering space. Prove that any map $f : \mathbf{RP}^2 \to S^1$ is homotopic to a constant map.

3. Let $p : E \longrightarrow B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

4. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{4k} \times \mathbf{RP}^2 \times \mathbf{RP}^{2n} \to \mathbf{CP}^{4k} \times \mathbf{RP}^2 \times \mathbf{RP}^{2n}$$

be a map. Prove that f always has a fixed point.

5. Define the Whitehead map $w : S^{n+k-1} \to S^n \vee S^k$. Prove that the element $[w] \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

6. Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

7. Compute the ring structure $H^*(\mathbf{RP}^{2n+1}; \mathbf{Z}/8)$.

8. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \le p, q \le n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

9. Let X be a finite simply-connected CW-complex with $\widetilde{H}_n(X) = 0$ for all n. Prove that X is contractible.

10. Let X be a CW-complex. Prove that the group $H^1(X; \mathbf{Z})$ is a free abelian group.

Qualifying Exam, Differential Geometry Fall 2005

Name: **ID# :**

There are 10 questions in this exam.

Problem 1. The graph of the function

$$f : (-\pi, \pi) \times (-\pi, \pi) \to \mathbb{R}^1 \qquad (x, y) \mapsto e^{x+y}$$

defines a smooth surface Σ^2 in Euclidean space $\mathbb{E}^3$.

a) Find an expression for the tangent plane to Σ^2 at the point $\begin{pmatrix} 1 \\ 1 \\ e^2 \end{pmatrix}$.

b) Calculate the components of the first fundamental form of Σ^2 with the coordinate basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$.

c) Write down an integral expression for the area of the surface Σ^2.

d) Let

$$\beta : [0, 1] \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

be a path in $\mathbb{R}^2$. Write down an explicit integral expression for the length of the path $\alpha(t) := \sigma \circ \beta(t)$, where $\sigma(x, y) = \begin{pmatrix} x \\ y \\ e^{x+y} \end{pmatrix}$ is the map whose image is Σ^2.

Problem 2. a) Given a smooth surface Σ^2 in Euclidean space $\mathbb{E}^3$. state the definition of the second fundamental form K.

b) Show that K is an $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor on Σ^2.

c) Show that K is symmetric, in the sense that $K(v, w) = K(w, v)$ for $v, w \in T_p\Sigma$.

Problem 3. Let M be a smooth manifold and let Σ be a subset of M.

a) Carefully state the conditions necessary for Σ to be an embedded submanifold.

b) Let $M = GL(n, \mathbb{R})$. Show that $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\}$ is an embedded submanifold of $GL(n, \mathbb{R})$.

c) Show that $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid A^T A = Id\}$ is an embedded submanifold of $GL(n, \mathbb{R})$.

Note: For

$$\begin{aligned} &\det : GL(n, \mathbb{R}) \to \mathbb{R} \\ &A \mapsto \text{determinant of } A, \\ &D(\det(A))B = \det A \cdot \operatorname{tr}[A^{-1}B]. \end{aligned}$$

For

$$\begin{aligned} &\mathrm{S} : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) \\ &A \mapsto A^T A, \\ &D(\mathrm{S}(A))B = B^T A + A^T B. \end{aligned}$$

Problem 4. Let $g = dx^2 + dy^2 + dz^2 + dw^2$ be a Riemannian metric on $\mathbb{R}^4$ and let $\Omega = dx \wedge dz + dy \wedge dw$ be a symplectic 2-form on $\mathbb{R}^4$. Let

$$i : \mathbb{R}^3 \to \mathbb{R}^4 \qquad (x, y, z) \mapsto (x, y, z, x^2 + y^2 + z^2)$$

be an embedded submanifold of $\mathbb{R}^4$.

a) Find the pullback of g to $i(\mathbb{R}^3)$. Is it a Riemannian metric? Explain.

b) Find the pullback of Ω to $i(\mathbb{R}^3)$. Is it a symplectic 2-form? Explain.

c) Write an integral expression for the volume of the hypersurface $i(\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\})$.

d) Let $H : \mathbb{R}^4 \to \mathbb{R}$ and let $\frac{d}{dt}\gamma = \Omega^{-1}(dH, \cdot)$, $\gamma(0) = p \in \mathbb{R}^4$ determine $\gamma : I \to \mathbb{R}^4$. Show that H is conserved along γ.

Problem 5. Recall the definition of the Lie derivative of a vector field W along a vector field V

$$\mathcal{L}_V W(p) = \lim_{h \to 0} \frac{(\theta_{-h})_* W_{\theta_h(p)} - W_p}{h}$$

for θ_h the flow of V.

a) For the case $V(p) \neq 0$, show that $(\mathcal{L}_V W)(p) = [V, W]_{(p)}$.

b) Show (using part a) that in terms of a local coordinate chart

$$\mathcal{L}_V W = V^a \frac{\partial}{\partial x^a} W^b - W^a \frac{\partial}{\partial x^a} V^b.$$

c) Verify that for 1-form α, $\mathcal{L}_V \alpha = i_V d\alpha + d i_V \alpha$.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let

$$g = e^{2f(x,y)}(dx^2 + dy^2)$$

be a Riemannian metric on $\mathbb{R}^2$. Compute the scalar curvature for the Levi-Civita (Riemannian) connection corresponding to g.

Problem 7. Let M be a smooth, oriented, compact n-dimensional manifold with boundary.

a) Given a careful statement of Stokes' Theorem for a smooth $(n-1)$-form ω on M.

b) Let $N := \mathbb{R}^3 \setminus \{0\}$. Define the following 2-form on N:

$$\alpha = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

b1) Show that $d\alpha = 0$.

b2) Show that $\int_{x^2+y^2+z^2=r^2} \alpha$ is independent of the constant r.

b3) Knowing that $\int_{x^2+y^2+z^2=1} \alpha = 4\pi$, compute $\int_{(x-1)^2+(y-2)^2+(z-3)^2=1} \alpha$ and $\int_{(x-1)^2+(y-2)^2+(z-3)^2=25} \alpha$.

b4) Show that there does not exist a 1-form β o N for which $d\beta = \alpha$.

Problem 8. Let ∇ be a connection on a Riemannian manifold (M, g) which is metric compatible and has nonzero torsion Q.

Find an expression for the Christoffel symbols

$$\Gamma^a{}_{bc} = dx^a(\nabla_{\frac{\partial}{\partial x^c}} \frac{\partial}{\partial x^b})$$

of this connection in terms of the component of the metric and their derivatives, and in terms of the components for the torsion $Q^a{}_{bc}$. (carefully show how you obtain this expression)

Problem 9. A homogeneous space consists of (M, G, ϕ) where M us a smooth manifold, G is a Lie group, and ϕ is a transitive action on M.

a) Show that

a1) $(S^{n-1}, SO(n), \phi =$ rotation action $)$ is a homogeneous space.

a2) $(\mathbb{R}^{2,+}, SL(2,\mathbb{R}), \hat{\phi})$ is a homogeneous space, for

$$\mathbb{R}^{2,+} = \{(x,y)\,|\,y>0\}$$
$$SL(2,\mathbb{R}) = \{A \in \mathbb{R}^{2\times 2}\,|\,\det A = 1\}$$
$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x,y)\right) = \left(\mathrm{Re}\left(\frac{a(x+iy)+b}{c(x+iy)+d}\right), \mathrm{Im}\left(\frac{a(x+iy)+b}{c(x+iy)+d}\right)\right).$$

b) Recall the theorem which states that if (M, G, ϕ) is a homogeneous space, then for any $p \in M$, there is a diffeomorphism from G/G_p to M, where G_p is the isotropy group for p. Use this to argue that $SO(3)/SO(2)$ is diffeomorphic to S^2.

Problem 10. State the Poincare lemma for the deRham cohomology, and prove it (in your proof, you may assume the homotopy equivalence theorem for deRham cohomology).