Solution Problem 1.1 Since X is non-zero, we can choose local coordinates $x=(x_1,...,x_n)$ so that $X=\partial_1^x$. We then have $\Phi_t^X: x\rightarrow x+te_1$. Since this is an isometry, $\partial_1^x(g_{ij})=0$. We also have g_{11} is constant. As the Levi-Civita connection is torsion free and Riemannian, we have:

$$g(\nabla_{\partial_1^x}\partial_1^x,\partial_i^x)=\Gamma_{11i}=\tfrac{1}{2}\{2\partial_1^x g_{1i}-\partial_i^x g_{11})=0\,.$$

Thus the curves $t\rightarrow x+te_1$ are geodesics as desired.

Let X be a left invariant vector field. The flow $\Phi_t^X: g\rightarrow g\cdot\exp(tX)$ is given by right multiplication, hence an isometry as the metric is right invariant. Since X is a left invariant vector field and since the metric is left invariant, X has constant length. Thus by the Lemma in question, the curves $t\rightarrow\exp(tX)$ are geodesics.

Solution Problem 1.2. We use the structure equations $\Gamma_{ijk}=\frac{1}{2}(\partial_j g_{ik}+\partial_i g_{jk}-\partial_k g_{ij})$ to see that

$$\begin{array}{ll}
g(\nabla_{\partial_r}\partial_r,\partial_r)=0, & g(\nabla_{\partial_r}\partial_r,\partial_\theta)=0,\\
g(\nabla_{\partial_r}\partial_\theta,\partial_r)=0, & g(\nabla_{\partial_r}\partial_\theta,\partial_\theta)=\cosh r\sinh r,\\
g(\nabla_{\partial_\theta}\partial_r,\partial_r)=0, & g(\nabla_{\partial_\theta}\partial_r,\partial_\theta)=\cosh r\sinh r,\\
g(\nabla_{\partial_\theta}\partial_\theta,\partial_r)=-\cosh r\sinh r, & g(\nabla_{\partial\theta}\partial_\theta,\partial_\theta)=0\,.
\end{array}$$

so

$$\begin{array}{ll}
\nabla_{\partial_r}\partial_r=0, & \nabla_{\partial_r}\partial_\theta=\coth r\partial_\theta,\\
\nabla_{\partial_\theta}\partial_r=\coth r\partial_\theta, & \nabla_{\partial_\theta}\partial_\theta=-\coth r\partial_\theta\,.
\end{array}$$

Since $\nabla_{\partial_r}\partial_r=0$ and since the ∂_r integral curves have constant length 1, the straight lines from the origin are constant speed geodesics; this is characteristic of geodesic coordinates.

To compute the curvature, we have to study

$$\begin{aligned}
&(\nabla_{\partial_\theta}\nabla_{\partial r}-\nabla_{\partial r}\nabla_{\partial_\theta}-\nabla_{[\partial_r,\partial_\theta]})\partial_r\\
=\;&-\nabla_{\partial_r}\nabla_{\partial_\theta}\partial_r=-\nabla_{\partial r}\{\coth r\partial_\theta\}\\
=\;&-\{\coth^2 r-\tfrac{1}{\sinh^r}\}\partial_\theta=\tfrac{\cosh^2 r-1}{\sinh^2 r}\partial_\theta=-\partial_\theta\,.
\end{aligned}$$

Consequently, we have

$$\tau=\frac{R(\partial_\theta,\partial_r,\partial_r,\partial_\theta)}{g_{11}g_{22}-g_{12}^2}=-\frac{g_{22}}{g_{22}}=-1\,.$$

This is the standard model of the hyperbolic space.

Solution Problem 1.3 Let Π be orthogonal projection from $\mathbb{R}^3$ to TM. Let ∇^e be the flat Euclidean connection on $\mathbb{R}^3$ and let ∇^M be the Levi-Civita connection of M. We wish to show that $\nabla^M=\Pi\circ\nabla^e$. To do this, we must establish that ∇^M is torsion free and Riemannian since there exists a unique torsion free Riemannian connection on M. Let X,Y,Z be vector fields on M. (In principle one should then extend these vector fields to $\mathbb{R}^3$ and then restrict back again. This is a bit of technical fuss I would not expect the students to go into). Since $\Pi[X,Y]=[X,Y]$, $\Pi Y=Y$, and $\Pi Z=\Pi Z$, we may use the properties that ∇^e has these properties to see:

$$\begin{aligned}
&\nabla_X^M Y-\nabla_Y^M X=\Pi(\nabla_X^e Y-\nabla_Y^e X)=\Pi[X,Y]=[X,Y],\\
&Xg(Y,Z)=g(\nabla_X^e Y,Z)+g(Y,\nabla_X^e Z)=g(\Pi\nabla_X^e Y,Z)+(Y,\Pi\nabla_X^e Z)=g(\nabla_X^M Y,Z)+g(Y,\nabla_X^Z)\,.
\end{aligned}$$

Solution Problem 1.4 The gradient of the defining relation $x^4+x^2+y^4+y^2+z^4+z^2=1$ is given by $(4x^3+2x,4y^3+2y,4z^3+2z)$. This vanishes when $4x^3+2x=0$, $4y^3+2y=0$, and $4z^3+2z=0$ which implies $x=y=z=0$ which is not on the surface. Thus by the implicit function theorem, the surface is in fact smooth. Note that necessarily $x\le 1$, $y\le 1$, $z\le 1$ so the surface is closed, bounded and hence compact. The map $T(x,y,z)=(-x,y,z)$ is an isometry of $\mathbb{R}^3$ which preserves the defining equation and hence which preserves the surface. The fixed point of an isometry is totally geodesic and in this case will be the 1 dimensional curve $x=0$, $y^4+y^2+z^4+z^2=1$. Thus the curves:

$$\begin{aligned}
C_1&:=\{(0,y,z):y^4+y^2+z^4+z^2=1\}\\
C_2&:=\{(x,0,z):x^4+x^2+z^4+z^2=1\}\\
C_3&:=\{(x,y,0):x^4+x^2+y^4+y^2=1\}
\end{aligned}$$

are closed geodesics in S. Similarly, consider $T(x,y,z) = (y,x,z)$. This is an isometry and the fixed point set gives curves

$$C_4 := \{(x,x,z) : 2x^4 + 2x^2 + z^4 + z^2 = 1\}$$
$$C_5 := \{(x,y,x) : 2x^4 + 2x^2 + y^4 + y^2 = 1\}$$
$$C_6 := \{(x,y,y) : x^4 + x^2 + 2y^4 + 2y^2 = 1\}$$

are closed geodesics in S. Finally, consider $T(x,y,z) = (-y,x,z)$. This is an isometry. Thus

$$C_7 := \{(x,-x,z) : 2x^4 + 2x^2 + z^4 + z^2 = 1\},$$
$$C_8 := \{(x,y,-x) : 2x^4 + 2x^2 + y^4 + y^2 = 1\},$$
$$C_9 := \{(x,y,-y) : x^4 + x^2 + 2y^4 + 2y^2 = 1\}$$

are closed geodesics.

Solution Problem 1.5 Consider instead the matrix group

$$\tilde{G} := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad+d \\ 0 & 1 \end{pmatrix}$$

this group is isomorphic to the $ax+b$ group. We therefore may define

$$e_1(g) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g \quad \text{and} \quad e_2(g) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g$$

as right invariant vector fields; since R_g is a linear map, $(R_g)_* = R_g$. We are using the canonical identification of $T\mathbb{R}^4 = \mathbb{R}^4$. Now comes a crucial point. We showed that if $X^r_\alpha(g) = \alpha \cdot g$ and if $X^r_\beta(g) = \beta \cdot g$, then $[X^r_\alpha, X^r_\beta] = -X^r_{[\alpha,\beta]}$. Thus $[e_1, e_2] = -e_2$.

You can also do this one directly. $R_{a,b}(x,y) = (ax, bx+y)$. Thus

$$(R_{a,b})_*(\partial_x) = a\partial_x + b\partial_y \quad \text{and} \quad (R_{a,b})_*(\partial_y) = \partial_y\,.$$

Thus in particular $e_2 := \partial_y$ is right invariant. Furthermore, if we set $e_1 = x\partial_x + y\partial_y$, then

$$(R_{a,b})_*(e_1(c,d)) = (R_{a,b})_*(c\partial_x + d\partial_y) = ac\partial_x + (bc+d)\partial_y = e_1(ac, bc+d) = e_1(R_{a,b}(c,d))$$

so e_1 also is right invariant. We may now compute $[e_1, e_2] = -e_2$.

Solution Problem 1.6 If $\mathbb{T}^3$ admitted a metric of positive sectional curvature, then Meyer's theorem would imply $\pi_1(\mathbb{T}^3)$ is finite. It is not.

Solution Problem 1.7 The results we are going to use are:

Theorem 3. (1) *If H is a closed subgroup of a Lie group G, then H is a Lie subgroup of G.*

(2) *If H is a Lie subgroup of a Lie group G, then the quotient space G/H inherits a natural smooth structure so that the projection $\pi : G \to G/H$ is a smooth fiber bundle with fiber H.*

(3) *If $A \subset B \subset C$ are a sequence of Lie groups where A is a closed subgroup of B is a closed subgroup of C, then the natural projection $\pi : C/A \to C/B$ is a smooth fiber bundle with fiber B/A.*

Let $\rho_k : O(n) \to Gr_k(n)$ be defined by setting $V_k(g) := span\{g_1, ..., g_k\}$ and let $\rho(g) := \{\rho_1(g) \subset \rho_2(g) \subset ...\}$ where $g = (g_1, ..., g_n)$ gives the columns of g. The Gram-Schmidt process shows ρ_k and ρ are surjective. Let H and H_0 be the isotropy subgroups at the basepoints $V_k^0 = span\{e_1, ..., e_k\}$ and

$$F^0 = \{\text{span}(e_1) \subset \text{span}(e_1, e_2) \subset ...\}$$

where $\{e_i\}$ is the standard base for $\mathbb{R}^k$. Then H and H_k are closed subgroups of a Lie group and hence Lie subgroups. Thus the quotient spaces $O(n)/H_k = Gr_k(n)$ and $O(n)/H = Flag(n)$ are smooth manifolds. Furthermore the natural projection $O(n)/H \to O(n)/H_k$ is a smooth fiber bundle. Note $H = O(k) \times O(n-k)$ and $H_k = \mathbb{Z}_2 \oplus ... \oplus \mathbb{Z}_2$.