1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2003

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ **SIGNATURE** ____________________

Please do any 10 problems out of the following 20.

1. Prove that any n-connected CW-complex is homotopy equivalent to a CW-complex with a single zero-dimensional cell and without cells of dimensions $1, 2, \ldots, n$.

2. State the Freudenthal Theorem. Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes of dimensions k, l respectively. Let $k + l + 1 < p$. Prove that the simplicial complexes K and L are not linked.

3. Let $p : E \longrightarrow B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

4. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{4k} \times \mathbf{RP}^2 \times \mathbf{RP}^4 \to \mathbf{CP}^{4k} \times \mathbf{RP}^2 \times \mathbf{RP}^4$$

be a map. Prove that f always has a fixed point.

5. Prove that the suspension $\Sigma(S^n \times S^k)$ is homotopy equivalent to the wedge $S^{n+1} \vee S^{k+1} \vee S^{n+k+1}$.

6. Define the Hopf invariant $h(\lambda)$ of an element $\lambda \in \pi_{4q-1}(S^{2q})$. Prove that $h(\lambda_1 + \lambda_2) = h(\lambda_1) + h(\lambda_2)$.

7. State the Poincaré Duality Theorem. Let M^3 be a closed connected orientable 3-manifold. Prove that $H_2(M^3; \mathbf{Z})$ has no torsion.

8. Let M_g^2 be oriented compact surface of genus g (i.e. M_g^2 is just the sphere with g holes/handles). Prove that there exists a map $f : M_g^2 \longrightarrow M_h^2$ such that $f_*[M_g^2] = [M_h^2]$ if and only if $g \geq h$.

9. Let X be a simply-connected CW-complex with $\widetilde{H}_n(X) = 0$ for all n. Prove that X is contractible.

10. Let $X = S^1 \times S^1$ and $Y = X/(S^1 \vee S^1) = S^2$. Let $f : X \to Y$ be the projection to the quotient space. Compute the homomorphisms $H_i(X) \to H_i(Y)$ and $\pi_i(X) \to \pi_i(Y)$ induced by f on the homotopy and homology groups for $i = 0, 1, 2$.

11. Recall that a *unitary* matrix is an $n \times n$ matrix X with complex entries such that $XX^* = I$ (where X^* is obtained from X by transposition and complex conjugation). Prove that the set U_n of all unitary $n \times n$ matrices is a Lie group and find its dimension.

12. Define the de Rham differential d of differential forms on manifolds. Show that for every differential 1-form ω and vector fields X, Y on a manifold M the following holds

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

13. Prove that for any manifold M its cotangent bundle T^*M is an orientable manifold.

14. Suppose that two vector fields X and Y on a manifold M are linearly independent at a point $p \in M$. Prove that X and Y can be simultaneously straightened at p (i.e. there exists local coordinate system $(x_1, x_2, \ldots)$ around p such that $X = \partial_{x_1}$ and $Y = \partial_{x_2}$) if and only if X and Y commute in some neighborhood of p.

15. The following is well known result in differential geometry:

Theorem 1. *Let g be a bi-invariant metric on a compact Lie group, then the exponential map defined using the Lie algebra of left invariant vector fields agrees with the exponential map defined by the geodesic flow of the metric.*

An essential ingedient in the derivation of this result was the following

Lemma 1. *Let Φ^X be the flow of a vector field X on a Riemannian manifold M. Assume that $g(X,X)$ is constant and that for a fixed t, the map Φ^X_t is an isometry. Then the integral curves for the vector field X are geodesics on M.*

Give a careful proof of Lemma 1. Then use Lemma 1 to give a careful proof of Theorem 1.

16. Let $x = r\cos\theta$ and $y = r\sin\theta$ be the usual polar coordinates on $\mathbf{R}^2$. Let g be a Riemannian metric on $\mathbf{R}^2$. Suppose that we may express $g = dr^2 + \sinh^2(r)d\theta^2$ for $r > 0$.

(1) Show that (x, y) are geodesic normal coordinates. Cite carefully any theorems that you use.

(2) Determine the scalar curvature of this metric. Justify carefully any steps in the computation.

17. Let $M \subset \mathbf{R}^3$ be a surface with the induced metric g. State and prove a result which relates the Levi-Civita connection of M to the Levi-Civita connection of $\mathbf{R}^3$.

18. Let M be a subset of $\mathbf{R}^3$ which given by the equation

$$x^4 + x^2 + y^4 + y^2 + z^4 + z^2 = 1.$$

Show that M is a smooth submanifold of $\mathbf{R}^3$ and that there are at least 9 distinct nontrivial closed geodesics on M.

19. Let G be the $ax+b$ group. More precisely, if $a > 0$ and if $b \in \mathbf{R}$, let $f_{a,b}(x) := ax + b$. Then $f_{a,b} \circ f_{c,d} = f_{ac,b+ad}$. This defines a group structure on $\{(a,b) \in \mathbf{R}^2 \mid a > 0\}$. Write down a basis $\{e_1, e_2\}$ for the Lie algebra of right invariant vector fields and determine $[e_1, e_2]$. Cite carefully any results that you use.

20. Show that $S^1 \times S^1 \times S^1$ does not admit a metric of positive sectional curvature.