Qualifying Exam in Algebraic Topology Fall 2002

You must show all your work. Carefully state any results which you use.

Problem #1. Let $S^n := \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ be the unit sphere. Let $\mathbb{Z}_2 = \{\pm 1\}$ act on S^n via scalar multiplication and let real projective space be the quotient: $\mathbb{RP}^n := S^n/\mathbb{Z}_2$. Let $\rho_n : S^n \to \mathbb{RP}^n$ be the natural projection. Show that $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\rho_{n-1}} D^n$.

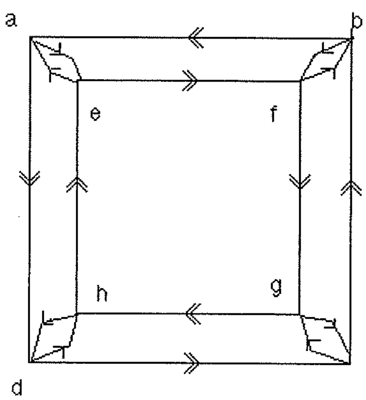
Problem #2. Let $SO(n)$ be the special orthogonal group. Determine $\pi_k(SO(n))$ for $k = 1, 2$ and $n \geq 3$. Give a careful statement of any theorems you use in your computations.

Problem #3. Let $\rho : E \to X$ be the covering of the figure 8 pictured on the right:

1) Is the cover a normal cover?

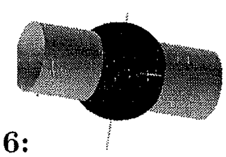
2) Is the deck group Abelian?

3) The fundamental group of E is the free group on n generators for what value of n?



Problem #4: Prove or disprove the following assertion: "Let X and Y be connected finite CW complexes. Suppose that for each k there is an isomorphism ϕ_k from $H_k(X)$ to $H_k(Y)$. Then X and Y have isomorphic homotopy groups."

Problem #5. Let $Y = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - .5)(x^2 + y^2 + z^2 - 1) = 0\}$. Find $H_*(Y)$.

Picture: 

Problem #6:
1) Show if X is the realization of a finite simplicial complex, $\pi_1(X)$ is finitely generated.
2) Show there exists a compact metric space Y so $\pi_1(Y)$ is not countable.

Problem #7. Let p be prime. Let $\mathbb{Z}_p$ act on $S^{2k-1} \subset \mathbb{C}$ by complex multiplication. Let $X := S^{2k-1}/\mathbb{Z}_p$ be the quotient space. An oracle informs you that X is the realization of a finite simplicial complex of dimension $2k - 1$ and that $H_q(X; \mathbb{Z}_p) = \mathbb{Z}_p$ for $q \leq 2k - 1$. Using this information, determine $H_*(X; \mathbb{Z})$.

Problem #8. Determine $Tor_n^{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$ and $Ext^n_{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$.

Problem #9. Let $X := \mathbb{RP}^2 \times \mathbb{RP}^4 \times \mathbb{RP}^6 \times \mathbb{RP}^8 \times \mathbb{RP}^{10}$. Show that any continuous map from X to itself has a fixed point.

Problem #10. Show that $S^2 \times S^3$ is not homotopy equivalent to $S^2 \vee S^3 \vee S^5$.

SOLUTIONS TO QUALIFYING EXAM IN ALGEBRAIC TOPOLOGY FALL 2002

Problem #1. Let $S^n := \{x \in \mathbb{R}^{n+1} : |x|^2\}$. Let $\mathbb{Z}_2$ act on S^n via scalar multiplication and let real projective space be the quotient $\mathbb{RP}^n := S^n/\mathbb{Z}_2$. Let $\rho_n : S^n \to \mathbb{RP}^n$ be the natural projection. Show that $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\rho_{n-1}} D^n$. (This gives a cell structure to $\mathbb{RP}^n$.)

Solution Problem #1. Let $\tilde{X} = \mathbb{RP}^{n-1} \cup_{\rho_{n-1}} D^n$. Identify D^n with the upper hemisphere in $\mathbb{RP}^n$. Let i be the natural inclusion of $\mathbb{RP}^{n-1}$ into $\mathbb{RP}^n$. Let $\tilde{\rho}_n$ be the restriction of ρ_n to D^n. Let $f := i \dot{\sqcup} \tilde{\rho}_n : \mathbb{RP}^{n-1} \dot{\sqcup} D^n \to \mathbb{RP}^n$. We then have that f is surjective since $\tilde{\rho}_n$ is surjective. Since f respects the glueing involved, it estends to a 1-1 onto map from $\tilde{f} : \tilde{X} \to \mathbb{RP}^n$. Since S^{n-1} is compact, $\mathbb{RP}^{n-1}$ is compact and hence, since $\mathbb{RP}^{n-1} \dot{\sqcup} D^n \to \tilde{X}$ is surjective, $\tilde{X}$ is compact. Thus $\tilde{f}$ will be a homeomorphism once we can argue $\mathbb{RP}^n$ is Hausdorff. The easiest way to do that is to let $\sigma : x \to (x_i x_j)$ define a map from S^n to $M_n(\mathbb{R})$; $\sigma(x)$ is simply orthogonal projection on x and is continuous. This induces a set theoretic map $\tilde{\sigma}x : \mathbb{RP}^n \to M_{n+1}(\mathbb{R})$ which thus extends continously. As S^n is compact, $\mathbb{RP}^n$ is compact. Since $\tilde{\sigma}$ is 1-1, it is a homeomorphism to its range. Thus $\mathbb{RP}^n$ is Hausdorff.

Problem #2. Let $SO(n)$ be the special orthogonal group. Determine $\pi_k(SO(n))$ for $k = 1, 2$ and $n \geq 3$. Give a careful statement of any theorems you use in your computations.

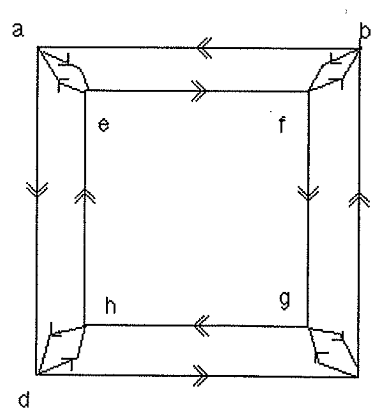
Solution Problem #2. We showed in class that $SO(3) = \mathbb{RP}^3 = S^3/\mathbb{Z}_2$. We showed that if $F \to E \to X$ is a covering projection, then $\pi_k(X) = \pi_k(E)$ for $k \geq 2$. Thus $\pi_2(SO(3)) = \pi_2(S^3) = 0$. Furthermore, since $\pi_1(S^3) = 0$, $\pi_1(SO(3)) = \mathbb{Z}_2$ is the deck group. We proceed inductively and assume $\pi_2(SO(k)) = 0$ and $\pi_1(SO(k)) = \mathbb{Z}_2$ for $k \geq 3$. We showed in class that $SO(k) \to SO(k+1) \to S^k$ is a twisted product. Hence has the Covering Homotopy Property with respect to the n cube - i.e. is a Serre fibration. We now use the long exact sequence of a fibration to see that:

$$\begin{aligned} &\pi_3(S^k) \to \pi_2(SO(k)) \to \pi_2(SO(k+1)) \to \\ &\pi_2(S^k) \to \pi_1(SO(k)) \to \pi_1(SO(k+1)) \to \pi_1(S^k) \end{aligned} \tag{1}$$

Since $k \geq 3$, $2 < k$ so $\pi_1(S^k) = 0$ and $\pi_2(S^k) = 0$. Since $\pi_2(SO(k)) = 0$, the inductive step follows since:

$$0 \to \pi_2(SO(k+1)) \to 0 \text{ and } 0 \to \pi_1(SO(k)) \to \pi_1(SO(k+1)) \to 0.$$

Problem #3. Let $\rho : E \to X$ be the covering of the figure 8 pictured on the right:



1) Is the cover a normal cover?

2) Is the deck group Abelian?

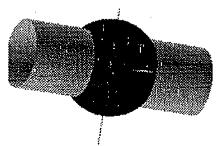
3) The fundamental group of E is the free group on n generators for what value of n?

Solution Problem #3. Define $T \in D$ to be the 90 degree clockwise rotation given by: $T = (abcd)(efgh)$ and $S \in D$ to interchange the inner and outer rings where we have $S = (bf)(ce)(hd)(ga)$. The deck group acts transitively of the vertices so the covering is normal. Since $TS \neq ST$, the group is non-Abelian. Take the plane with 9 points removed. This deformation retracts to the indicated diagram and also deformation retracts to a boquet of 9 circles. Thus the fundamental group of the diagram is the free group on 9 generators.

Problem #4: Prove or disprove the following assertion: "Let X and Y be connected finite CW complexes. Suppose that for each k there is an isomorphism ϕ_k from $H_k(X)$ to $H_k(Y)$. Then X and Y have isomorphic homotopy groups."

Solution Problem #4. This is false. Let $X = S^1 \vee S^1 \vee S^2$ and $Y = S^1 \times S^1$. Then these have homology groups $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}$, $H_2 = \mathbb{Z}$, $H_k = 0$ for $k \geq 3$. On the other hand S^2 is a retract of X so $\pi_2(X) \neq 0$. Since $\mathbb{R}^2$ covers Y, $\pi_2(Y) = 0$.

Problem #5. Let $Y = \{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2-.5)(x^2+y^2+z^2-1) = 0\}$. Find $H_*(Y)$.

Picture: 

Problem #5. sphere intersect a cylinder. Let X be the surface of the sphere of radius 1 about the origin union a cylinder of radius .5 about the z axis. By fattening things up we can create an open cover $\mathcal{O}_1$, $\mathcal{O}_2$ so $\mathcal{O}_1 \downarrow S^2$, $\mathcal{O}_2 \downarrow S^1$, and $\mathcal{O}_\infty \cap \mathcal{O}_2 \downarrow S^1 \sqcup S^1$. Thus the interesting part of Mayer-Vietoris becomes:

$$0 \to H_2(S^2) = \mathbb{Z} \to H_2(X) \to H_1(S^1 \dot\sqcup S^1) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_2)^1_*} H_1(S^1) = \mathbb{Z} \to H_1(X)$$
$$\to H_0(S^1 \dot\sqcup S^1) \to H_0(S^2) \oplus H_0(S^1) \to H_0(X) \to 0.$$

Now each circle of $S^1 \dot\sqcup S^1$ maps via a homotopy equivalence to the cylinder. Thus $(i_2)^1_*$ is surjective and the sequence decouples to be

$$0 \to \mathbb{Z} \to H_2(X) \to \mathbb{Z} \to 0$$
$$0 \to H_1(X) \to 2\mathbb{Z} \to 2\mathbb{Z} \to \mathbb{Z} \to 0.$$

Thus we conclude

$$H_k(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem #6:.
1) Show if X is the realization of a finite simplicial complex, $\pi_1(X)$ is finitely generated.
2) Show there exists a compact metric space Y so $\pi_1(Y)$ is not countable.

Problem #6. We may assume without loss of generality that X is connected. For each vertex V of X, let γ_V be a path from the basepoint to V. For each oriented edge of the form $e := (V_1, V_2)$, let $\sigma_e = \gamma_{V_1} * (V_1, V_2) * \gamma_{V_2}^{-1}$ be a closed loop in X. The simplicial approximation theorem shows any closed loop is homotopic to a loop which can be written in the form $(V_0, V_1) * (V_1, V_2) * ...$ i.e. a product of oriented edges. Rewriting this, up to homotopy, in the form $(V_0, V_1) * \gamma_{V_1}^{-1} * \gamma_{V_1} * (V_2, V_3) * ...$ then shows that the paths $\{\sigma_e\}$ as e ranges over the oriented edges generates $\pi_1(X)$; thus $\pi_1(X)$ is finitely generated.

Let C_n be the circle of radius $\frac{1}{n}$ about the point $(\frac{1}{n}, 0) \in \mathbb{R}^2$. Let $X = \cup_{n\geq 1} C_n$ - this is a closed compact subset of $\mathbb{R}^2$ with basepoint 0. We show that $\pi_1(X, *)$ is uncountable as follows. Let $I_n = [\frac{1}{n+1}, \frac{1}{n}]$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...)$ be an infinite sequence where $\varepsilon_i = 0, 1$. Define a map $f_\varepsilon : [0, 1] \to X$ as follows. Set $f_\varepsilon(0) = 0$. Let $t \in I_n$. If $\varepsilon_n = 0$, set $f_\varepsilon(t) = 0$. If $\varepsilon_n = 1$, let f_ε wrap I_n around the circle C_n once with $f_\varepsilon(\frac{1}{n}) = f_\varepsilon(\frac{1}{n+1}) = 0$. Clearly f_ε is continuous on each compact set I_n and the definition is consistent on $I_n \cap I_{n+1}$ since $f_\varepsilon(\frac{1}{n+1}) = 0$. Finally, since the circles C_n have shrinking radii, f_ε is continuous at 0. We define a retract r_n to be the identity on C_n and to map C_k to 0 for $k \neq 0$. Suppose f_ε and f_ϱ are fixed endpoint homotopic. Then $[r_n f_\varepsilon] = [r_n f_\varrho]$ as elements of $\pi_1(C_n, 0)$. This implies $\varepsilon_n = \varrho_n$ for all n and hence $\varepsilon = \varrho$. Since there are an uncountable number of such sequences, we conclude $\pi_1(X, 0)$ is uncountable.

Problem #7. Let p be prime. Let $\mathbb{Z}_p$ act on $S^{2k-1} \subset \mathbb{C}$ by complex multiplication. Let $X := S^{2k-1}/\mathbb{Z}_p$ be the quotient space. An oracle informs you that X is the realization of a finite simplicial complex of dimension $2k - 1$ and that $H_q(X; \mathbb{Z}_p) = \mathbb{Z}_p$ for $q \leq 2k - 1$. Using this information, determine $H_*(X; \mathbb{Z})$.

Solution Problem #7. Let $\mathbb{Z}_p$ act on $S^{2k-1} \subset \mathbb{C}$ by complex multiplication. Consider the quotient space $X := S^{2k-1}/\mathbb{Z}_p$. An oracle informs you that X is the realization of a finite simplicial complex and that $H_q(X; \mathbb{Z}_p) = \mathbb{Z}_p$ for $q \leq 2k - 1$. Using this information, determine $H_*(X; \mathbb{Z})$. SOLUTION: Since $\mathbb{Z}_p$ is a finite group acting without fixed points on a compact metric space, the map $\pi : S^{2k-1} \to X$ is a covering projection. Let π_* and π^* denote transfer and induction. Then $\pi_* \pi^*$ is multiplication by p. Since $H_q(S^{2k-1}; \mathbb{Z}) = 0$ for $0 < q < 2k-1$, this means $pH_q(X; \mathbb{Z}) = 0$. Since X is the realization of a finite simplicial complex, $H_q(X; \mathbb{Z})$ is a finitely generated Abelian group. Thus we have $H_q(X; \mathbb{Z}) = \nu_q \cdot \mathbb{Z}_p$. Furthermore $H_{2k-1}(X; \mathbb{Z})$ is a free $\mathbb{Z}$ module so $H_{2k-1}(X; \mathbb{Z}) = \nu_{2k-1}\mathbb{Z}$. We have for dimensional reasons $H_q(X; \mathbb{Z}) = 0$ for $q > 2k - 1$. Since X is are connected, $H_0(X; \mathbb{Z}) = 0$. We use the Universal coefficient theorem therefore to compute

$$H_q(X; \mathbb{Z}_p) = H_q(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \oplus Tor(H_{q-1}(X; \mathbb{Z}), \mathbb{Z}_p)$$

This yields the relations: $\nu_1 = 1$, $\nu_q + \nu_{q-1} = 1$ for $2 \leq q \leq 2k - 1$. Consequently $\nu_q = 1$ for $1 \leq q \leq 2k - 1$ and q odd while $\nu_q = 0$ for q even. Thus $H_0(X; \mathbb{Z}) = H_{2k-1}(X; \mathbb{Z}) = \mathbb{Z}$, $H_q(X; \mathbb{Z}) = \mathbb{Z}_p$ for $1 \leq q < 2k - 1$ and q odd, $H_q(X; \mathbb{Z}) = 0$ otherwise.

Problem #8. Determine $Tor_n^{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$ and $Ext^n_{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$.

Solution Problem #8. Determine $Tor_n^{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$ and $Ext^n_{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4)$. SOLUTION: Take a projective resolution of $\mathbb{Z}_8$ of the form

$$...\mathbb{Z}_{16} \xrightarrow{8} \mathbb{Z}_{16} \xrightarrow{2} \mathbb{Z}_{16} ... \mathbb{Z}_{16} \xrightarrow{8} \mathbb{Z}_{16} \to \mathbb{Z}_8 \to 0.$$

Tensoring with $\mathbb{Z}_4$ then yields $...\mathbb{Z}_4 \xrightarrow{8} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 ... \mathbb{Z}_4 \xrightarrow{4} \mathbb{Z}_4 \to 0$. From this we get that $Tor_n^{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4) = \mathbb{Z}_4$ if $n = 0$ while $Tor_n^{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4) = \mathbb{Z}_2$for $n \geq 1$. Applying the Hom functor gives $\mathbb{Z}_4 \xrightarrow{8^*} \mathbb{Z}_4 \xrightarrow{2^*} \mathbb{Z}_4 \xrightarrow{8^*} \mathbb{Z}_4 ....$ and hence $Ext^n_{\mathbb{Z}_{16}}(\mathbb{Z}_8, \mathbb{Z}_4) = \mathbb{Z}_4$ for $n = 0$ and $\mathbb{Z}_2$ for $n \geq 1$.

Problem #9. Let $X := \mathbb{RP}^2 \times \mathbb{RP}^4 \times \mathbb{RP}^6 \times \mathbb{RP}^8 \times \mathbb{RP}^{10}$. Show that any continuous map from X to itself has a fixed point.

Solution Problem #9. We have $H_k(\mathbb{RP}^{2n}; \mathbb{Q}) = 0$ for $k > 0$ while $H_0(\mathbb{RP}^{2n}; \mathbb{Q}) = 0$. Thus applying the Kunneth formula recursively yields $H_*(X : \mathbb{Q}) = 0$ for $* > 0$ while $H_0(X; \mathbb{Q}) = \mathbb{Q}$. Thus $\mathcal{L}(f) = 1$. Since $\mathbb{CP}^n$ has the structure of a finite CW complex, X has the structure of a finite CW complex and the Lefschetz theorem applies to show f has a fixed point.

Problem #10. Show that $S^2 \times S^3$ is not homotopy equivalent to $S^2 \vee S^3 \vee S^5$.

Solution Problem #10. Show that $S^2 \times S^3$ is not homotopy equivalent to $S^2 \vee S^3 \vee S^5$. Answer. The Kunneth formula shows $H^*(S^2 \times S^3) = H^*(S^2) \otimes H^*(S^3)$ is a ring isomorphism. Thus, in particular, $\alpha_2 \cup \alpha_3 \neq 0$ i.e. there is a non-trivial multiplication. On the other hand if $\rho_\nu : S^2 \vee S^3 \vee S^5 \to S^\nu$ is the natural retract and j_ν the natural inclusion of $S^\nu \subset S^2 \vee S^3 \vee S^5$ for $\nu = 2, 3, 5$, and if x_ν are the generators of the non-zero cohomology groups, then $x_\nu = \rho_\nu^*(\alpha_\nu)$ and $i_\nu^*(x_\mu) = \delta_{\nu\mu}\alpha_\nu$ where α_ν is the generator of $H^\nu(S^\nu)$. Then $x_2 \cup x_3 = cx_5$ so applying i_5^* we get $c\alpha_5 = i_5^*(cx_5) = i_5^*(x_2) \cup i_5^*(x_3) = 0$ so the ring structure here is trivial.