TOPOLOGY-GEOMETRY QUALIFYING EXAM, FALL 2000

Do 10 problems.

Algebraic Topology:

(1) Prove that $\mathbf{C}P^n$ is a CW-complex, and compute its integral homology groups.
(2) Show that the map $O(n) \to S^{n-1}$ that takes a matrix to its top row is a fiber bundle. In what range is the inclusion $O(n-1) \to O(n)$ an isomorphism on homotopy groups? What is the dimension of $O(n)$ as a manifold?
(3) Identify the space $\mathbf{H}P^1$ and prove that $\mathbf{Z}$ is a subgroup of $\pi_7 S^4$.
(4) State the Whitehead Theorem. Is $S^1 \times S^1 \simeq S^1 \vee S^1 \vee S^2$? Is $S(S^1 \times S^1) \simeq S^2 \vee S^2 \vee S^3$?
(5) Suppose that $H_i(X)$ is all torsion for $i > 0$ and X is a connected simplicial complex. Prove that X has the fixed-point property. (All maps from X to itself have a fixed point.) Must $X \times X$ also have the fixed point property?
(6) Use acyclic models to prove that if $f, g : \Delta_*(X) \to \Delta_*(X)$ are natural maps of chain complexes that agree on Δ_0 then f is chain homotopic to g.
(7) Calculate the degree of $x \mapsto -x$ on the space S^n.
(8) Give an example of a covering space $X \xrightarrow{p} Y$ and a map $f : X \to X$ so that $p \circ f = p$ but f isn't a homeomorphism. Prove this can't happen if p is a finite covering.
(9) Is it possible for X and Y to both be pointed spaces that are not contractible, but such that $X \wedge Y$ is contractible?
(10) Calculate $Ext(\mathbf{Z}/(m), \mathbf{Z})$ and $Ext(\mathbf{Z}/(m), \mathbf{Z}/(n))$.

Differential Geometry:

(11) Let $\gamma(s) := (x(s), y(s), 0)$ be a smooth curve in $\mathbb{R}^2$ which is parametrized by arc length. Assume $y(s) > 0$ for all s. Revolve γ around the x axis to form a surface of revolution which can be parametrized in the form $T(u, s) := (x(s), \cos(u)y(s), \sin(u)y(s))$. For which values of s are the curves $u \to T(u, s)$ geodesics?
(12) Let ∇ be the Levi-Civita connection of a Riemannian metric. Let

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

be the associated curvature operator. State and derive the curvature symmetries which define the space of algebraic curvature tensors.
(13) Prove there exists a bi-invariant volume form on any compact connected Lie group.

(14) Let M be a smooth connected manifold which admits a complete metric g of non-positive sectional curvature. Show that M does not admit a complete metric h with $\text{Ric}_h(\xi, \xi) \geq |\xi|^2$.

(15) Let G be a Lie group. Let $X \in g$ be a left invariant vectcor field. Show that the integral curves for X exist for all time.

(16) Let (M, g) be a 2 dimensional Riemannian manifold. Prove or disprove the following assertion: "There exists a local orthonormal frame $\{e_1, e_2\}$ for the tangent bundle of M so that $[e_1, e_2] = 0$".

(17) (a) Let σ be a geodesic. Give a careful definition of what it means for Y to be a Jacobi vector field along σ.

(b) Let S^2 be the unit sphere in $\mathbb{R}^3$. Let $\sigma(t) := (\cos(t), \sin(t), 0)$ be the equator in S^2. Describe all the Jacobi vector fields along σ.

(18) Let L be a complex line bundle.

(a) Let ∇ be a connection on L. Relative to a local non-zero section to L, define the curvature and Chern form $c_1(\nabla)$.

(b) Show that $c_1(\nabla)$ is a closed 2 form which is independent of the local section chosen.

(c) Show the DeRham cohomology class of $c_1(\nabla)$ is independent of the connection L. (d) Give an example of a complex line bundle L so that $[c_1(\nabla)]$ is non-zero in de Rham cohomology. Your example should be correct, but you need not prove that it works.

(19) Let G be a compact connected Lie group.

(a) Define the coalgebra structure on $H^*_{DeR}(G)$ making $H^*_{Der}(G)$ into a Hopf algebra.

(b) State the Lerray structure theorem in this context.

(c) State the cohomology ring structure of $H^*(U(n))$.

(20) Let $ds^2 = \frac{dx^2+dy^2}{y^2}$ on the upper half plane. Let $\gamma(t) = (t, 1)$. Let $e(t)$ be a vector field which is parallel translated along γ with $e(0) = (0, 1)$. Let $\theta(t)$ be the angle $e(t)$ makes with $\dot{\gamma}$. Determine $\theta(t)$.

QUALIFYING EXAM, FALL 2000, ALGEBRAIC TOPOLOGY SOLUTIONS

(1) Given a homeomorphism $X^{(2n-2)} \to \mathbf{C}P^{n-1}$ where $X^{(2n-2)}$ is a CW-complex, we take an attaching map $f_n : S^{2n-1} \to X^{(2n-2)} = \mathbf{C}P^{n-1}$ which is the usual quotient map:

$$(z_0, \dots, z_{n-1}) \mapsto [z_0 : \dots : z_{n-1}]$$

We now need to construct a homeomorphism

$$g : X^{(2n)} = X^{(2n-2)} \cup D^{2n} \to \mathbf{C}P^n.$$

We map the $X^{(2n-2)} = \mathbf{C}P^{n-1}$ by

$$[z_0 : \dots : z_{n-1}] \mapsto [z_0 : \dots : z_{n-1} : 0]$$

and the D^{2n} by

$$(z_0, \dots, z_{n-1}) \mapsto [z_0; \dots : z_{n-1} : \sqrt{1 - |z_1|^2 - \dots - |z_{n-1}|^2}].$$

Note that these two maps are compatible under the identifications, and are each continuous separately, so are continuous from the quotient space $X^{(2n)}$.

Finally check that the map is one to one and onto. To see onto, first note that if we make the last coordinate of $\mathbf{C}P^n$ 0, we get exactly the image of $\mathbf{C}P^{n-1}$ under the first part of the description of g. If the last coordinate isn't 0, normalize so that it is a scalar and so that the homogeneous coordinates $[z_0 : \dots : z_{n-1} : r]$ have norm 1. Then this point is the image of $(z_0, \dots, z_{n-1}) \in D^{2n}$.

To see one to one, notice that $g|_{\mathbf{C}P^{n-1}}$ is one to one on the $\mathbf{C}P^{n-1}$. It follows that the map is one to one on the subspace $\mathbf{C}P^{n-1}$, which is also the quotient of $S^{2n-1} \subseteq D^{2n}$. The interior of the disk is taken to the complement of $\mathbf{C}P^{n-1}$ sitting in $\mathbf{C}P^n$, so it suffices to check that the map is one to one on the interior.

If $(z_0, \dots, z_{n-1}), (w_0, \dots, w_{n-1}) \in \mathrm{Int}(D^{2n})$ but their images are the same under g, then

$$(z_0, \dots, z_{n-1}, \sqrt{1 - \sum |z_i|^2}) = \lambda(w_0, \dots, w_{n-1}, \sqrt{1 - \sum |w_i|^2}).$$

Since both points were in the interior of the disk, the last coordinates are non-zero, and then λ must be real and positive. So

$$(z_0, \dots, z_{n-1}) = r(w_0, \dots, w_{n-1}).$$

If $r < 1$, then $\sqrt{1 - \sum |z_i|^2} < \sqrt{1 - \sum |w_i|^2}$. But at the same time,

$$\sqrt{1 - \sum |w_i|^2} = r\sqrt{1 - \sum |z_i|^2} < \sqrt{1 - \sum |z_i|^2}.$$

This is a contradiction. One gets a similar contradiction if $r > 1$. So $r = 1$ and $(z_0, \dots, z_{n-1}) = (w_0, \dots, w_{n-1})$.

Finally, we note that since there are no adjacent dimensions with cells in them, the cellular chain map is 0, so the homology is generated by the cells, and is thus $\mathbf{Z}$ in even dimensions less than or equal to $2n$, and 0 else.

(2) Given an $A \in O(n)$, I'll write x for $p(A)$.

Let U be the set of points where the first coordinate is not 0. Assume A is in $p^{-1}U$. (A similar construction applies if U is the set of points where some other coordinate is not 0.) Let e_i be the vector with a 1 in the ith position and 0s elsewhere. We define an element of $O(n)$ as follows:

- Take the linearly independent vectors $\{x, e_2, \ldots, e_n\}$. (These are linearly independent since x is non-zero in the first coordinate.)
- Apply Gram-Schmidt to get an orthonormal set $\{x, v_2, \ldots, v_n\}$.
- Make a matrix by taking x for the top row, and v_i for the ith row.
- Call this $\phi(x)$.

Since p and Gram-Schmidt are continuous, ϕ is continuous.

Now we note that

$$A\phi(x)^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & b_{11} & \ldots & b_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n-1,1} & \ldots & b_{n-1,n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

Denote the map that sends A to $B \in O(n-1)$ by Φ. Φ is continuous on $p^{-1}U$ since ϕ, matrix inversion, and matrix multiplication (and projection to a sub-matrix) are continuous.

We now have a map

$$p^{-1}U \xrightarrow{(p,\Phi)} U \times O(n-1).$$

It is fairly straightforward to construct an inverse map.

$$(x, B) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \phi(x)$$

By the long exact sequence in homotopy groups of a fibration, the map $O(n-1) \to O(n)$ is an isomorphism for $i \leq n-2$.

$O(n-1)$ has dimension $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$.

(3) $\mathbf{H}P^1$ is isomorphic to S^4. We map $S^4 \to \mathbf{H}P^1$ by

$$(x, y, z, w, t) \mapsto [(x, y, z, w), (t, 0, 0, 0)].$$

This is a bijection, and since S^4 is compact and $\mathbf{H}P^1$ is Hausdorff, it is a homeomorphism.

Now we have a fibration $S^7 \to \mathbf{H}P^1$, and the inverse image of $[(1,0,0,0),(0,0,0,0)]$ is S^3. Using the long exact sequenc of that fibration, we get

$$\pi_7 S^3 \to \pi_7 S^7 \to \pi_7 S^4 \to \pi_6 S^3.$$

Since the map $S^3 \to S^7$ is homotopic to the constant map, we get

$$0 \to \mathbf{Z} \to \pi_7 S^4 \to \pi^6 S^3.$$

(4) The two spaces are not the same. $\pi_2 S^1 \times S^1 = 0$, but $\pi_2 S^2$ is a summand of $\pi_2(S^1 \vee S^1 \vee S^2)$.

To produce a homotopy equivalence

$$S(S^1 \times S^1) \to S^2 \vee S^2 \vee S^3$$

we "add" using the structure of the domain as a suspension the maps $S\pi_1$, $S\pi_2$ and $S\eta$ where the π_i are the projections to the factors, and η is the projection to the top cell of the product.

The sum of these maps is a homology isomorphism, hence a homotopy equivalence by the Whitehead theorem.

(5) We use the Lefschetz theorem for homology with coefficients in the field $\mathbf{Q}$. By the universal coefficient theorem, $H_i(X;\mathbf{Q}) = \mathbf{Q}$ when $i = 0$ and 0 otherwise. So the Lefschetz number of *any* map is 1, hence the map has a fixed point.

Combining the Künneth Theorem and the Universal coefficient theorem, we see that $X \times X$ has the same rational homology, so also has the fixed point property.

(6) Define D to be 0 on Δ_0.

Suppose $D : \Delta_k(X) \to \Delta_{k+1}(X)$ is defined for all $k < n$ so that

i $D\delta + \delta D = f - g$ on elements of $\Delta_k(X)$ and

ii D is natural.

We wish to define $D(\iota_n) \in \Delta_{n+1}(\Delta_n)$ (recall that the space Δ_n is the standard n-simplex in $\mathbf{R}^{n+1}$).

Observe that $z = f(\iota_n) - g(\iota_n) - D(\partial\iota_n)$ satisfies

$$\partial z = f(\partial\iota_n) - g(\partial\iota_n) - f(\partial\iota_n) + g(\partial\iota_n) = 0$$

(the critical observation here is that

$$\delta D\delta = f\delta - g\delta - D\delta\delta = f\delta - g\delta).$$

So z is a cycle in $\Delta_n(\Delta_n)$. Since Δ_n is an acyclic space, it follows that $z = \partial c$. Define $D(\iota_n) = c$.

Now if σ is an arbitrary element of $\Delta_n(X)$, $\sigma = \sigma_\Delta(\iota_n)$. Define $D(\sigma) = \sigma_\Delta(c)$. Then it is clear that Δ is a natural transformation in dimension n.

$$D\partial(\sigma) + \partial D(\sigma) = D \circ \sigma_\Delta(\partial\iota_n) + \partial\sigma_\Delta(D(\iota_n)) =$$

$$\sigma_\Delta(D\partial + \partial D)(\iota_n) = \sigma_\Delta(f - g)(\iota_n) = f(\sigma) - g(\sigma).$$

This verifies the chain homotopy condition.

(7) Let $f_n : S^n \to S^n$ be the map that reverses the *first* coordinate:

$$f_n(x_0, \ldots, x_n) = (-x_0, x_1, \ldots, x_n).$$

Then $f_{0*} : \tilde{H}_0(S^0) \to \tilde{H}_0(S^0)$ is multiplication by -1.

By induction if f_r has degree -1, then the commutativity of the following diagram tells us that f_{r+1} also has degree -1:

$$\begin{array}{ccccccc}
\tilde{H}_{r+1}(S^{r+1}) & \xrightarrow[\cong]{} & H_{r+1}(S^{r+1}, D_+^{r+1}) & \xleftarrow[\cong]{} & H_{r+1}(D_-^{r+1}, S^r) & \xrightarrow[\delta]{} & \tilde{H}_r(S^r) \\
\downarrow f_{(r+1)*} & & \downarrow f_{(r+1)*} & & \downarrow f_{(r+1)*} & & \downarrow f_{r*} \\
\tilde{H}_{r+1}(S^{r+1}) & \xrightarrow[\cong]{} & H_{r+1}(S^{r+1}, D_+^{r+1}) & \xleftarrow[\cong]{} & H_{r+1}(D_-^{r+1}, S^r) & \xrightarrow[\delta]{} & \tilde{H}_r(S^r)
\end{array}$$

By symmetry this works in any coordinate.

The antipodal map is the composition of $n + 1$ maps, each of which reverses one coordinate. So it has degree $(-1)^{n+1}$.

(8) We take Y to be the figure 8, and X to be a cover which to the right looks like the "infinite TV antenna" and to the left looks like the negative x-axis with a loop at each integer. The map f is defined to be the map $X \to X$ that take $(0, 0)$ to $(-1, 0)$; i.e. translates one unit to the left.

Now suppose $p : X \to Y$ is a covering space with $f : X \to X$ and $pf = p$, with finite fibers. Let x_0 be a basepoint of X, and $x_1 = f(x_0)$. Since f exists, we get $p_{\#}\pi_1(X, x_0) \subseteq p_{\#}\pi_1(X, x_1)$. Since it is a finite cover, the subgroup must have index 1, so the two groups are the same. This implies the existence of a $g : X \to X$ with $pg = p$ and $g(x_1) = x_0$. By uniqueness then, $fg = 1_X$ and $gf = 1_X$.

(9) Yes, and even CW-complexes. Let $X = S^3 \cup_2 S^4$ and $Y = S^3 \cup_3 S^4$. Then $X \wedge Y$ is a CW-complex with bottom cell in dimension 6, so is simply connected.

A calculation with the Künneth theorem shows that $\tilde{H}_*(X \wedge Y) = 0$, so by the Whitehead theorem the space is contractible. But neither of X and Y are contractible since they have non-zero homology groups.

(10) We take a resolution

$$0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathbf{Z}/(m) \to 0$$

and take homomorphisms to $\mathbf{Z}$.

This gives the exact sequence

$$0 \leftarrow Ext(\mathbf{Z}/(m), \mathbf{Z}) \leftarrow \mathbf{Z} \xleftarrow{\cdot m} \mathbf{Z} \leftarrow Hom(\mathbf{Z}/(m), \mathbf{Z}) \leftarrow 0.$$

This gives $Ext(\mathbf{Z}/(m), \mathbf{Z}) \cong \mathbf{Z}/(m)$.

To calculate $Ext(\mathbf{Z}/(m), \mathbf{Z}/(n))$, we take homomorphism of the same exact sequence into $\mathbf{Z}/(n)$.

$$0 \leftarrow Ext(\mathbf{Z}/(m), \mathbf{Z}/(n)) \leftarrow \mathbf{Z}/(n) \xleftarrow{\cdot m} \mathbf{Z}/(n) \leftarrow Hom(\mathbf{Z}/(m), \mathbf{Z}/(n)) \leftarrow 0.$$

So $Ext(\mathbf{Z}/(m), \mathbf{Z}/(n)) = (\mathbf{Z}/(m))/(n) = \mathbf{Z}/(d)$ where d is the greatest common divisor of m and n.

Qualifying Exam Solutions for Differential Geometry Fall 2000

Solution to Problem #1. Let ${}^e\nabla$ be the Euclidean connection. Let $U := T_*\partial_u$ and $S := T_*(\partial_s)$. We compute:

$$\begin{aligned}
&S = (\dot{x}(s), \cos(u)\dot{y}(s), \sin(u)\dot{y}(s))\\
&U = (0, -\sin(u)y(s), \cos(u)y(s))\\
&{}^e\nabla_U U = (0, -\cos(u)y(s), -\sin(u)y(s))\\
&(S, {}^e\nabla_U U) = y(s)\dot{y}(s)\\
&(U, {}^e\nabla_U U) = 0
\end{aligned}$$

Now the Levi-Civita connection is the projection of the Euclidean connection to the surface. Thus ${}^M\nabla_U U = 0$ if and only if ${}^e\nabla_U U$ is perpendicular to both U and to S or equivalently if and only if $y(s)\dot{y}(s) = 0$ i.e. $\dot{y}(s) = 0$. Thus the curves in question are geodesics if and only if $\dot{y} = 0$.

Solution to Problem #2. The Levi-Civita connection is characterized by the properties

$$\begin{aligned}
&(\nabla_X Y, Z) + (Y, \nabla_X Z) = X(Y, Z) \text{ (Riemannian)}\\
&\nabla_X Y - \nabla_Y X = [X, Y] \text{ (Torsion Free)}
\end{aligned}$$

The three curvature symmetries are

$$\begin{aligned}
&(R(X,Y)Z, W) + (Z, R(X,Y)W) = 0 \text{ (Skew Symmetric)}\\
&(R(X,Y)Z, W) = (R(Z,W)X, Y)\\
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \text{ (Bianchi Identity)}
\end{aligned}$$

We prove the first identity by computing:

$$\begin{aligned}
(R(X,Y)Z, W) &= (\nabla_X\nabla_Y Z, W) - (\nabla_Y\nabla_X Z, W) - (\nabla_{[X,Y]}Z, W)\\
&= X(\nabla_Y Z, W) - Y(\nabla_X Z, W) - [X,Y](Z,W)\\
&\quad - (\nabla_Y Z, \nabla_X W) + (\nabla_X Z, \nabla_Y W) + (Z, \nabla_{[X,Y]}W)\\
&= XY(Z,W) - YX(Z,W) - [X,Y](Z,W)\\
&\quad - X(Z, \nabla_Y W) + Y(Z, \nabla_X W)\\
&\quad - Y(Z, \nabla_X W) + X(Z, \nabla_Y W)\\
&\quad - (Z, \nabla_Y\nabla_X W) + (Z, \nabla_X\nabla_Y W) + (Z, \nabla_{[X,Y]}W)\\
&= -(Z, R(X,Y)W).
\end{aligned}$$

ETC. If you want me to, I will type out the computation but it is in any text on the subject and my fingers are getting tired. More intelligently is to first prove there exist coordinate systems so $g_{ij}(P) = \delta_{ij}$ and $\partial_k g_{ij}(P) = 0$. One then has

$$R_{ijkl}(P) = \frac{1}{2}(g_{jl/ik} + g_{ik/il} - g_{jk/il} - g_{il/jk})(P);$$

the curvature symmetries then cited above follow immediately.

Solution Problem #3. Let gbe the Lie algebra of G; this is the finite dimensional vector space of left invariant vector fields on G. Choose any basis e_i for g and define a Riemannian metric on G by setting $(e_i, e_j) = \delta_i j$. Let $\nu = e_1 \wedge \ldots$. This is a left invariant volume form on G. Note that $\int_G \nu \neq 0$. If $\nu = d\alpha$, then $\int_G \nu = 0$ by Stokes theorem. Thus $[\nu]$ is non-trivial in de Rham cohomology. Let $\gamma(t)$ be any curve in G with $\gamma(0) = 1$ and let $R(t)$ be right multiplication by γ. Then $R(t)^*\nu$ is again a left invariant volume form since right and left multiplication commute. Thus $R(t)^*\nu = c(t)\nu$. On the other hand, the homotopy axiom for de Rham cohomology yields $[\nu] = [R(t)^*\nu]$. Since $[\nu] \neq 0$, we have $c(t) = 1$. Thus $R(t)^*\nu = \nu$ and ν is again right invariant.

Solution Problem #4. Assume M admits a complete metric of positive Ricci curvature. Then M is compact and the fundamental group of M is finite. Assume M admits a complete metric of negative sectional curvature. Then there are no conjugate points. Thus the exponential map is a local diffeomorphism and in fact is a covering projection. Since M is compact, the fundamental group, which is the inverse image of the basepoint, is infinite. This is a contradiction.

Solution Problem #5. Let $\gamma(t)$ be an integral curve for X such that $\gamma(0) = 1$ and which is defined for $t \in (-\epsilon, \epsilon)$. If $h \in G$, let $\gamma_h(t) = \gamma(t) \cdot h$. By naturality, γ_h is an integral curve for $(R(h))_*X = X$ starting at h. Thus the integral curves exist for time ϵ uniformly on G. But now they can be patched together to be defined uniformly for all time on G.

Solution Problem #6. This is definitely false. The Christoffel symbols of the Levi-Civita connection can be expressed in terms of the Lie bracket. Were it possible to construct a torsion free orthonormal frame, the Christoffel symbols and hence the curvature would vanishe identically. It doesn't.

Solution Problem #7. We say that Y is a Jacobi vector field along a geodesic σ if Y satisfies the differential equation $\ddot{Y} + R(Y, X)X = 0$. Let $\sigma(t) = (\cos(t), \sin(t), 0)$ be the equatorial vector field. Let $U(t) := \dot{\sigma}(t) = (-\sin(t), \cos(t), 0)$ and $V(t) = (0, 0, 1)$; these are parallel vector fields along σ. We $R(V, U)U = V$ since S^2 has constant sectional curvature +1. Let $Y(t) = (a + bt)U + (c\cos(t) + d\sin(t))V$. This satisfies the differential equation involved. Since $Y(0) = aU + cV$ and $\dot{Y}(0) = bU + dV$, this is the most general possible Jacobi vector field; a Jacobi vector field is determined by its initial values and the initial values of the derivative.

Problem #8. Let L be a complex line bundle. A connection ∇ on L is a linear map $\nabla : C^\infty(L) \to T^*M \otimes L$ so that $\nabla(f\vec{s}) = df \otimes \vec{s} + f\nabla\vec{s}$. If $\vec{s}$ is non-zero, we define $\omega_{\vec{s}}$ by the identity $\nabla\vec{s} = \omega_{\vec{s}} \otimes \vec{s}$. We define $\Omega_{\vec{s}} = d\omega_{\vec{s}}$. This is clearly a closed 2 form. We set $c_1(\nabla, \vec{s}) = \frac{i}{2\pi}\Omega_{\vec{s}}$.

If $\vec{t} = g\vec{s}$, then

$$\omega_{\vec{t}}\vec{t} = \nabla(g\vec{s}) = dg\vec{s} + g\nabla\vec{s} = (dg \cdot g^{-1} + \omega_{\vec{s}})g\vec{s} = (d\log g + \omega_{\vec{s}})\vec{t}$$

so that $\omega_{\vec{t}} = d\log(g) + \omega_{\vec{s}}$ and consequently $\Omega_t = \Omega_s$ and $c_1(\nabla)$ is defined independent of the particular local section chosen. Let ∇ and $\tilde{\nabla}$ be two connections. Then $(\nabla - \tilde{\nabla})(f\vec{s}) = f(\nabla - \tilde{\nabla})\vec{s}$ since the df terms cancel. Thus $(\nabla - \tilde{\nabla})\vec{s} = \alpha \otimes \vec{s}$ yields an invariantly defined 1 form α. Since $\omega - \tilde{\omega} = \alpha$, we have $c_1(\nabla) - c_1(\tilde{\nabla}) = \frac{1}{2\pi}d\alpha$ vanishes in de Rham cohomology.
Example The Hopf line bundle over $\mathbb{CP}^n$.

Example Let e_i be complex 2×2 Clifford matrices over S^2; these are self-adjoint complex matrices so $e_i e_j + e_j e_i = 2\delta_{ij}$. Let $e(x) = x_0 e_0 + x_1 e_1 + x_2 e_2$ and let L be the $+1$ eigenbundle. Then $c_1(L) \neq 0$.
Example Let $\mathbb{T}^2 = [0, 2\pi] \times S^1$ where we glue $(0, \lambda)$ to $(2\pi, \lambda)$. Let $L = [0, 2\pi] \times S^1 \times \mathbb{C}$ where we glue $(0, \lambda, z)$ to $(2\pi, \lambda, \lambda z)$. This gives a bundle over $\mathbb{T}^2$ where $c_1(L) \neq 0$. There are other examples.

Solutions Problem #9. Let $m : G \times G \to G$ be the multiplication in the group. Then $m^* : H^*_{DeR}(G) \to H^*_{Der}(G \times G)$ is a ring morphism. The Kunneth formula in DeRham cohomology expressed in terms of the Hodge decomposition theorem shows $H^*_{Der}(G \times G) = H^*_{Der}(G) \otimes H^*_{Der}(G)$. Thus m^* makes $H^*_{Der}(G)$ into a Hopf algebra. Consequently we have $H^*_{Der}(G) = \Lambda(y_{k_1}, ..., y_{k_r})$ is an exterior on a finite number of generators in odd degrees. (More generally, one would get $C[x_{l_1}, ..., x_{l_r}] \otimes \Lambda(y_{k_1}, ..., y_{k_s})$ but as the cohomology is finitely generated there are no polynomial generators. The unitary group has $H^*_{Der}(U(k)) = \Lambda[y_1, ..., y_{2k-1}]$.

Solution Problem #10. Let $ds^2 = \frac{dx^2 + dy^2}{y^2}$ on the upper half plane. Let $\gamma(t) = (t, 1)$. Let $X = \partial_x$ and $Y = \partial_Y$. We compute:

$$
\begin{aligned}
&(\nabla_X X, X) = \frac{1}{2} X(X, X) = 0 \\
&(\nabla_X X, Y) = -(X, \nabla_X Y) = -(X, \nabla_Y X) = -\frac{1}{2} Y(X, X) = y^{-3} \\
&(\nabla_X Y, X) = (\nabla_Y X, X) = \frac{1}{2} Y(X, X) = -y^{-3} \\
&(\nabla_X Y, Y) = \frac{1}{2} X(Y, Y) = 0 \\
&\nabla_X X = y^{-1} Y \text{ and } \nabla_X Y = -y^{-1} X
\end{aligned}
$$

We set $Y = 1$. Let $U(t) = \sin(t)X + \cos(t)Y$ along γ. Then $\nabla_X U = (\cos(t)X - \sin(t)Y) + (\sin(t)Y - \cos(t)X) = 0$. Thus, modulo a possible sign convention, the parallel vector field turns at a constant angle t relative to the horizontal.