THE NONCOMMUTATIVE ALGEBRAIC GEOMETRY OF QUANTUM PROJECTIVE SPACES

by

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AS-regular algebras of global dimension n were defined in [3] as non-commutative analogues of the polynomial algebra on n-variables. An AS-regular algebra A is a finitely generated connected $\mathbb{N}$-graded k-algebra. It has a presentation as $T(V^*)/I$ where $T(V^*)$ is the tensor algebra on a finite-dimensional vector space V^* and I is a homogeneous ideal. The zeroes of the generators of I define a projective scheme Γ called the point scheme of A. The AS-regular algebras of global dimension 4 have not been classified. We construct many new families of AS-regular algebras of global dimension 4 whose point schemes Γ have finitely many points. We prove that all of our examples are finitely generated modules over their centers and are Noetherian domains. For one of our examples we find the fat point modules. Finally we prove results about generic k-algebras with 5 generators and 10 quadratic relations.

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DEDICATION

For John Albert Bowers and Frederick James Goetz.

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CHAPTER I

INTRODUCTION

I.1. Background and History

Non-commutative projective algebraic geometry studies a non-commutative graded algebra A by associating a category $\operatorname{Proj} A$ in which one can do geometry. More specifically, $\operatorname{Proj} A$ is a quotient category of the module category $\operatorname{Gr}\operatorname{Mod} A$ of graded modules by the dense subcategory of direct limits of finite-dimensional modules. This is not a new idea. In the 50's, Serre taught us that the projective algebraic geometry of a commutative graded ring R is the study of a quotient category $\operatorname{Proj} R$ of the graded module category $\operatorname{Gr}\operatorname{Mod} R$. More precisely, let R be a commutative connected $\mathbb{N}$-graded k-algebra, where k is an algebraically closed field, and assume R is generated by R_1 as a k-algebra. Let $(X, \mathcal{O}_X) = \operatorname{Proj} R$ be the projective scheme defined by R. We define an equivalence relation on graded R-modules by $M \sim M'$ if there is an integer n for which $M_{\geq n} \cong M'_{\geq n}$ where $M_{\geq n} = \bigoplus_{d \geq n} M_d$.

Theorem I.1.1. *[10] [Ex. II.5.9] The category of finitely generated R-modules modulo the equivalence relation $\sim$ is equivalent to the category of coherent $\mathcal{O}_X$-modules.*

Therefore the geometric object $\operatorname{Proj} R$ can be studied via algebraic objects, the graded modules of R.

In the case of a non-commutative algebra A there is no associated geometric scheme X. However the category $\operatorname{Proj} A$ exists, and should be considered as the space in which the geometry of A lives. This was the point of view taken by Artin and Schelter in [2] where they defined AS-regular algebras.

Definition I.1.2. Let k be a field. An AS-regular algebra A is a connected $\mathbb{N}$-graded k-algebra which is Gorenstein, has finite global dimension, and finite Gelfand-Kirillov dimension.

The polynomial ring $k[x_1, \ldots, x_n]$ is an AS-regular algebra of global dimension n. We think of AS-regular algebras as noncommutative deformations of $k[x_1, \ldots, x_n]$. We now discuss some of the highlights and successes of the geometric point of view in the study of AS-regular algebras.

The AS-regular algebras of global dimensions 1 and 2 are trivial to classify. The AS algebras of dimension 3 which are generated in degree 1 were classified in [2, 3]. Stephenson classified the AS-algebras of dimension 3 which are not necessarily generated in degree 1 in [28, 29, 30].

The classification is based on the following ideas. Given an AS-regular algebra A, the relations of A can be thought of as functions on a product of projective spaces. Then the zeroes of the relations are a commutative projective scheme. This scheme is denoted by Γ. The Gorenstein condition is used to prove that Γ is the graph of an automorphism τ of a projective scheme E. The pairs (E, τ) are then used to classify the AS-regular algebras of dimension 3. The point scheme of A is the pair (E, τ) or equivalently the scheme Γ. For example, in the global dimension 3 case when A is generated in degree 1, generically E is an elliptic curve in $\mathbb{P}^2$.

Using the above classification, Artin, Schelter, Tate, van den Bergh et. al., proved the following remarkable theorems.

Theorem I.1.3. *[3] Suppose A is an AS-regular algebra of global dimenson 3 which is generated in degree 1, then A is a Noetherian domain.*

Theorem I.1.4. *[4] Let A be an AS-regular algebra of global dimension 3 which is generated in degree 1, and let (E, τ) be the point scheme. Then τ has finite order if and only if A is a finitely generated module over its center.*

Despite the numerous examples that have been studied, a classification in the global dimension 4 case still seems to be a long way off. In this thesis we restrict ourselves to studying quantum $\mathbb{P}^3$'s. Recall, the Hilbert series of a graded k-algebra $A = \bigoplus_{n\geq 0} A_n$ is $H_A(t) = \sum_{n\geq 0} \dim_k A_n$.

Definition I.1.5. A quantum $\mathbb{P}^n$ is an AS-regular algebra of global dimension n whose Hilbert series is $\frac{1}{(1-t)^n}$.

So a quantum $\mathbb{P}^3$ has the same Hilbert series as the polynomial ring $k[x_1, x_2, x_3, x_4]$. Many examples of quantum $\mathbb{P}^3$'s have been studied. For example, Cassidy [7] has classified the quantum $\mathbb{P}^3$'s which are normal extensions of global dimension 3 AS-regular algebras. Another example is the Sklyanin algebras, first defined by Odesskii and Feigen, which are quantum $\mathbb{P}^n$'s constructed from the data of an elliptic curve

E and an automorphism τ of E. The Sklyanin algebras of dimension 4 have been extensively studied in [14, 23, 25, 26]. The survey [24] is a good summary of what is known about the 4-dimensional Sklyanin algebras. Shelton and Vancliff have studied in [21] the quantum $\mathbb{P}^3$'s which have as quotients twisted homogeneous coordinate rings of quadric surfaces in $\mathbb{P}^3$. As a final example, LeBruyn in [12] has studied Clifford algebras.

As a result of studying these examples, people have isolated the following classes of modules as important in the study of AS-regular algebras. These modules should be thought of as geometric objects, hence the terminology. This point of view is supported by the fact that in many situations, the modules defined below are parametrized by projective *commutative* schemes.

Definition I.1.6. Let A be a quantum $\mathbb{P}^n$.

1) A point module P is a graded right A-module which is cyclic, generated in degree 0, and has Hilbert series $\dfrac{1}{1-t}$.

2) A fat point module F is a graded right A-module which has GKdim 1, is GK-critical and has multiplicity $m > 1$, i.e., $\dim_k F_n = m$ for all $n >> 0$.

3) A line module L is a graded right A-module which is cyclic, generated in degree 0, and has Hilbert series $\dfrac{1}{(1-t)^2}$.

In [18], Shelton and Vancliff prove there are projective schemes representing the functors of point modules and line modules. In fact the point scheme (E, τ)

parametrizes the point modules. In the quantum $\mathbb{P}^3$ case, generically, the point scheme Γ consists of 20 points counted with multiplicity. In particular, generically Γ is a zero-dimensional scheme. In [18], Shelton and Vancliff prove the counter-intuitive result: if $\dim(\Gamma) = 0$ then the relations of A are precisely the 2-forms which vanish on Γ.

The examples studied above: extensions of AS-regular algebras, Sklyanin algebras, AS-regular algebras containing a quadric surface, all have point schemes of dimension at least one. So, while interesting, they should not be considered as generic representatives of the class of quantum $\mathbb{P}^3$'s. We make the following definition.

Definition I.1.7. Let A be a quantum $\mathbb{P}^3$. We say A is a generic quantum $\mathbb{P}^3$ if $\dim \Gamma = 0$.

I.2. Statements of Main Theorems

The thesis is organized as follows. Chapter 2 begins with general definitions and preliminaries, we discuss the result II.1.6 and how it can be used to construct examples of quantum $\mathbb{P}^n$'s. Then we prove Theorem II.2.4 which is useful for proving that a given quantum $\mathbb{P}^3$ is finite over its center.

Theorem I.2.1. *Let A be a finitely generated $\mathbb{N}$-graded k-algebra with $H_A(t) = \dfrac{p(t)}{(1-t)^n}$ for some $p(t) \in k[t]$ and $n \in \mathbb{N}$. Suppose that $a_1, \dots, a_n$ is a regular sequence of homogeneous elements of positive degree which are all central. Let $A' =$*

$A/(a_1, \ldots, a_n)$ and $Z(A)$ denote the center of A. Then:

1. *A is a finitely generated module over $Z(A)$.*

2. *$C := k[a_1, \ldots a_n]$ is a weighted polynomial ring.*

3. *A' is a finite-dimensional ring. A is a free C-module of rank $\dim_k A'$.*

Chapter 3 is the heart of the thesis. In III.1, we study an example of a quantum $\mathbb{P}^3$, A, found by Shelton and Tingey, whose point scheme has exactly 20 points. We compute the graded automorphism group (modulo scalars) of A. We compute the right graded twists of A. The main theorem of this section is Theorem III.1.20 where we prove that the Shelton-Tingey algebra and all of its right graded twists are finite modules over their centers.

In III.2, we use II.1.6 to construct many new examples of generic quantum $\mathbb{P}^3$'s. In Construction 1 we build an Ore extension of the first homogenized Weyl algebra $A_1(k)$. From this we construct a one parameter family of generic quantum $\mathbb{P}^3$'s, denoted by S_α, $\alpha \in k^*$. We refer the reader to III.2 for the definition of S_α. The main result is Theorem III.2.7:

Theorem I.2.2. *Let S_α be as in III.2.5. Let $C = k[x_1x_4 + x_4x_1, x_3^2, x_1x_2 + x_2x_1, x_1^2]$ and let Z denote the center of S_α. Then:*

1. *S_α is a finitely generated module over Z.*

2. *C is a weighted polynomial ring.*

3. S_α is a free module of rank 16 over C.

In fact, the dependence on the parameter α is illusory. We show that $S_\alpha \cong S_{\alpha'}$ for all $\alpha, \alpha' \in k^*$.

In Construction 2 we build families of generic quantum $\mathbb{P}^3$'s. We start from a skew polynomial ring and construct an Ore extension by a nontrivial derivation δ. From this we build some families of generic quantum $\mathbb{P}^3$'s. The main results are Examples III.2.11, and III.2.12, in which we demonstrate that each algebra in the family is a generic quantum $\mathbb{P}^3$ and is a finitely generated module over its center. We also determine the number of points in the point scheme for each of the examples, and for Example III.2.12 we calculate the multiplicities of each of the points in Γ.

Chapter 4 concerns the geometry of fat point modules and the incidence relations between point modules and line modules. We begin with general remarks on the notions of fat points and fat point modules. In Section 2 we study the geometry of the fat point modules for the twist of the Shelton-Tingey example, A^{d_3}. By III.1.20, A^{d_3} is a free module over a polynomial subring C. Given a fat point module F, let $\zeta = C \cap \mathrm{Ann}_A(F)$. We prove ζ is a closed point of $\mathrm{Proj}\, C$. For each closed point $\zeta \in \mathrm{Proj}\, C$ we construct functorially a finite-dimensional algebra A_ζ. We consider a basic affine subscheme $X \subset C$ and for each $\zeta \in X$ we study the algebra A_ζ. There are two main results. In Theorem IV.1.9 we determine the fat point modules of multiplicity 2 for which $\zeta \in X$. We refer the reader to IV.1.9 for the undefined terms.

Theorem I.2.3. *Let A denote the twist of the Shelton-Tingey example by the auto-*

morphism d_3*. Let* $C = k[x_1^2, x_2^2, \Omega_1, \Omega_2]$ *denote the central subalgebra given above. Let* X *denote the affine scheme of* $\operatorname{Proj} C$ *given by*

$$X = \{(\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2) \mid \alpha, \beta, \gamma \in k\}$$

. Then:

1) Generically, there exist two families of multiplicity 2 fat point modules. These families are given by two affine curves, C_1*,* C_2 *in* X*.*

2) For each point $\zeta \in C_i$*,* $i = 1, 2$ *there are exactly 2 non-isomorphic multiplicity 2 fat point modules lying over* ζ*.*

3) The truncated shift functor, $[-1]_{\geq 0}$*, on graded modules has order 2 on* C_1 *but has order 1 on* C_2*.*

The interpretation of 3) is as follows. If $F \in \mathcal{F}_1$ lies over ζ then $F[-1]_{\geq 0} \in \mathcal{F}_1$ and $F[-1]_{\geq 0}$ also lies over ζ. Whereas if $F' \in \mathcal{F}_3$ then $F'[-1]_{\geq 0} \cong F'$ as graded A-modules.

In Theorem IV.1.12 we find a three parameter family of multiplicity 4 fat point modules. For fixed values of the three parameters we show that there are precisely 2 non-isomorphic fat point modules. This shows that for generic $\zeta \in X$, the ring A_ζ is semisimple and is isomorphic to $M_4(k) \times M_4(k)$.

In Section 4 we find the incidence relations between the point modules and the line modules for the Shelton-Tingey example. Let P be a point module and L be a line module. We say P lies on L if there exists a surjective homomorphism of graded modules $L \twoheadrightarrow P$. For a point module P, let $\mathcal{L}_P$ be the scheme of line modules passing through P. Then we prove Theorem IV.2.5 which determines the scheme $\mathcal{L}_P$ for each of the 20 point modules for the Shelton-Tingey example.

Finally in Chapter 5, we start by proving Theorem V.1.2:

Theorem I.2.4. *The generic k-algebra on 5 linear generators and 10 quadratic relations has no truncated point or truncated line modules of length 3.*

From this it immediately follows that a generic k-algebra on 5 linear generators and 10 quadratic relations has no point or line modules. We define d-linear modules as in [17]. We then prove some results about the d-linear modules of a quantum $\mathbb{P}^4$.

CHAPTER II

PRELIMINARIES AND DEFINITIONS

II.1. Preliminaries

We start this chapter by introducing the definitions and notation which will be in force for the rest of the paper. Then we will discuss a certain recipe from [17] which can be used to produce new examples of quantum $\mathbb{P}^3$'s from the data of a given quantum $\mathbb{P}^3$ and a quadratic normal regular sequence.

Unless otherwise stated, k, will always denote an algebraically closed field of characteristic 0. A graded k-algebra A is understood to be an $\mathbb{N}$-graded k-algebra, $A = \bigoplus_{n \geq 0} A_n$, where $\dim_k A_n < \infty$ for all $n \geq 0$. All modules, unless otherwise stated, are graded right A-modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ with $\dim_k M_n < \infty$ for all $n \in \mathbb{Z}$. We say M is bounded below if there is an $n \in \mathbb{Z}$ for which $M_i = 0$ for all $i < n$. The shift of M by degree $i \in \mathbb{Z}$ is the module $M[i]$ which is M as a vector space but $M[i]_n = M_{i+n}$ for all $n \in \mathbb{Z}$. We will sometimes use the phrase "M is finite over A" to mean that M is a finitely generated A-module.

By $\operatorname{Gr}\operatorname{Mod} A$ we mean the category of graded right A-modules with morphisms given by degree 0 homomorphisms. The subcategory $\operatorname{Fdim} A$ is the full subcategory of $\operatorname{Gr}\operatorname{Mod} A$ consisting of direct limits of finite-dimensional modules. Then $\operatorname{Fdim} A$

is a dense subcategory [22] and following [5], we let $\operatorname{Proj} A = \operatorname{Gr}\operatorname{Mod} A/\operatorname{Fdim} A$ denote the corresponding quotient category. For $M, M' \in \operatorname{Gr}\operatorname{Mod}$, write $M \sim M'$ if $M_{\geq n} \cong M_{\geq n}$ for some $n \in \mathbb{Z}$. Then M and M' define the same object in $\operatorname{Proj} A$ if and only if $M \sim M'$.

Suppose A is a finitely generated k-algebra, and M is a finitely generated A-module. Let V be a finite-dimensional generating subspace of A which contains 1. Let F be a finite-dimensional subspace of M which generates M as an A-module. Define a function $d_M : \mathbb{N} \to \mathbb{N}$ by $d(n) = \dim_k(FV^n)$. One can show that d is independent of the choice of V and F, for example see [11]. Then the Gelfand-Kirillov dimension of M is given by

$$\operatorname{GKdim} M = \limsup \log_n d_M(n).$$

A graded, finitely generated A-module, M, of GKdim $= n$ is GK-homogeneous if every nonzero submodule $N \subset M$ has $\operatorname{GKdim} N = n$. M is GK-critical if every proper quotient module Q has GKdim $< n$. The Hilbert series of a graded module M is given by $H_M(t) = \sum_{n\in\mathbb{Z}} \dim_k M_n t^n$. If $H_M(t) = \frac{p(t)}{\prod_{i=1}^{n}(1-t^{d_i})}$ with $p(t) \in K[t]$ and $d_i \in \mathbb{N}$ then $\operatorname{GKdim} M$ is the order of the pole of $H_M(t)$ at $t = 1$.

Artin and Schelter defined the following class of rings in [2]. These will be the main object of study in the sequel.

Definition II.1.1. Let A be an $\mathbb{N}$-graded connected k-algebra with k a field. Then A is a regular algebra of dimension n if the following hold:

1) A has finite global dimension n

2) A has finite Gelfand-Kirillov dimension

3) A has the Gorenstein property.

We follow the definitions in [15]. Given a right R-module M, the projective dimension of M, $\operatorname{pd} M$, is the shortest length n of a finite projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M,$$

or ∞ if no finite resolution exists. The (right) global dimension of a ring R is given by

$$\operatorname{gldim} R = \sup\{\operatorname{pd} M\},$$

where the supremum is taken over all R-modules, M.

A connected graded k-algebra A of global dimension n satisfies the Gorenstein property if the trivial module $k_A = A/A_+$, ($A_+ = \bigoplus_{n>0} A_n$) has a projective resolution $P_\bullet$ by finitely generated projectives,

$$0 \to P_n \to \cdots \to P_0 \to k_A$$

for which the dual sequence $\operatorname{Hom}_{\operatorname{Gr\,Mod}}(P_\bullet, A)$,

$$0 \to P_0^* \to \cdots \to P_n^* \to_A k[e]$$

is a resolution of the trivial module ${}_A k[e]$ for some integer e. Equivalently $\operatorname{Ext}_A^p(k_A, A) = \delta_n^p k[e]$ for some integer e.

One can restrict to the class of regular algebras that have the same Hilbert series as polynomial rings. These are the quantum $\mathbb{P}^n$'s.

Definition II.1.2. Let A be an $\mathbb{N}$-graded k-algebra with k a field. Then A is a quantum $\mathbb{P}^n$ if the following hold:

1) A has finite global dimension $n+1$

2) $H_A(t) = \dfrac{1}{(1-t)^{n+1}}$

3) A has the Gorenstein property.

Definition II.1.3. Let R be a ring. Then we say $x \in R$ is normal if $xR = Rx$. We say $x \in R$ is regular if x is not a left or right zero divisor, and $x \in R$ is left (right) regular if x is not a left (right) zero divisor.

Notice that if $x \in R$ is normal and regular then x defines an automorphism $\Omega : R \to R$ via the rule $xr = \Omega(r)x$ for all $r \in R$.

Definition II.1.4. Let R be a ring. A sequence $x_1, \ldots, x_n$ is a normal regular sequence in R if:

1) x_1 is a normal, regular element in R

2) x_i is a normal, regular element in $R/(x_1, \ldots, x_{i-1})R$ for all $2 \leq i \leq n$.

Normal sequences are defined similarly. A useful way of detecting regular sequences in graded rings is the following.

Lemma II.1.5. *Let A be a locally finite graded k-algebra. Let $x_1, \ldots, x_n$ be a normal sequence of homogeneous elements with* $\deg(x_i) = d_i$. *Then $x_1, \ldots, x_n$ is a regular* *sequence if and only if*

$$H_{A/(x_1,\ldots,x_n)A}(t) = \prod_{i=1}^{n}(1 - t^{d_i}) \cdot H_A(t).$$

Proof. Use induction on n. For the case $n = 1$, regularity of x is equivalent to the sequence

$$0 \to A \xrightarrow{\cdot x} A \to A/xA \to 0$$

being exact. By counting dimensions, this is equivalent to $H_{A/xA}(t) = (1 - t^d)H_A(t)$, where $d = \deg x$. Now suppose $x_1, \ldots, x_n, x_{n+1}$ is a normal regular sequence. Then the sequence

$$0 \to A/(x_1, \ldots x_n)A \xrightarrow{\cdot x_{n+1}} A/(x_1, \ldots x_n)A \to A/(x_1, \ldots x_{n+1})A \to 0$$

is exact. So that

$$\begin{aligned} H_{A/(x_1,\ldots x_{n+1})A} &= (1 - t^{d_{n+1}})H_{A/(x_1,\ldots x_n)A} \\ &= \prod_{i=1}^{n+1}(1 - t^{d_i}) \cdot H_A(t) \qquad \text{by induction.} \end{aligned}$$

Conversely, if $x_1, \ldots, x_{n+1}$ is not regular then there is an index i for which $x_1, \ldots, x_{i-1}$ is regular and x_i is not regular modulo $x_1, \ldots, x_{i-1}$. Hence there is an exact sequence

$$0 \to K \to A/(x_1, \ldots x_{i-1})A \xrightarrow{\cdot x_i} A/(x_1, \ldots x_{i-1})A \to A/(x_1, \ldots x_i)A \to 0$$

with $K \neq 0$. Hence

$$\begin{aligned} H_{A/(x_1,\ldots,x_i)A} &> H_{A/(x_1,\ldots,x_i)A} - H_K \\ &= (1-t^{d_i})H_{A/(x_1,\ldots,x_{i-1})A} \\ &= \prod_{j=1}^{i}(1-t^{d_j}) \cdot H_A \qquad \text{by induction.} \end{aligned}$$

The notation $\sum_{n\geq 0} a_n t^n > \sum_{n\geq 0} b_n t^n$, for power series in $\mathbb{Z}[[t]]$, means that $a_n \geq b_n$ for all $n \in \mathbb{N}$ and for at least one $i \in \mathbb{N}$, $a_i > b_i$.

Now quotienting out the next element x_{i+1} we have the exact sequence

$$0 \to K' \to A/(x_1,\ldots x_i)A \xrightarrow{\cdot x_{i+1}} A/(x_1,\ldots x_i)A \to A/(x_1,\ldots x_{i+1})A \to 0$$

And again

$$\begin{aligned} H_{A/(x_1,\ldots,x_{i+1})A} &\geq H_{A/(x_1,\ldots,x_{i+1})A} - H'_K \\ &= (1-t^{d_{i+1}})H_{A/(x_1,\ldots,x_i)A} \\ &> \prod_{j=1}^{i+1}(1-t^{d_j}) \cdot H_A \end{aligned}$$

Therefore

$$H_{A/(x_1\ldots,x_{n+1})A} > \prod_{j=1}^{n+1}(1-t_j^d) \cdot H_A$$

as desired. □

The following theorem, due to Shelton-Tingey [17], gives a recipe for constructing new examples of quantum $\mathbb{P}^n$'s. Recall that given a quadratic algebra $A = T(V^*)/I$, its Koszul dual $A^!$ is $T(V^*)/I^\perp$ where $I^\perp$ is generated by $I_2^\perp \subset V^* \otimes V^*$. We define $I_2^\perp$ by fixing a nondegenerate bilinear form on $V^* \otimes V^*$.

Theorem II.1.6. *[17] Let A be a quantum $\mathbb{P}^n$ and suppose $x_1, \ldots, x_{n+1}$ is a quadratic normal regular sequence in A. If $B = (A/(x_1, \ldots, x_{n+1}))^!$ then B is a quantum $\mathbb{P}^n$.*

We now aim toward defining a "generic" 4-dimensional algebra. Let $A = T(V^*)/I$ where V^* is a 4-dimensional k-vector space, $T(V^*)$ is the tensor algebra on V^*, and I is a homogeneous ideal generated by quadratic elements, say $f_1, \ldots, f_6 \in V^* \otimes V^*$. Consider the scheme

$$\Gamma = \{p \in \mathbb{P}(V) \times \mathbb{P}(V) \mid f_i(p) = 0,\ 1 \leq i \leq 6\}.$$

Equivalently, consider the Segre embedding of $\mathbb{P}(V) \times \mathbb{P}(V)$ in $\mathbb{P}(V \otimes V)$ and let S be the associated homogeneous coordinate ring of $\mathbb{P}(V) \times \mathbb{P}(V)$. The quadratic relations, f_i, define (1,1)-homogeneous forms in S. Let J be the ideal of S generated by these relations. Then $\Gamma = \text{Proj}(S/J)$.

Define the *point scheme* of A to be the scheme Γ. By [21], for any quadratic $\mathbb{N}$-graded algebra having 4 generators and 6 quadratic relations, Γ is nonempty and generically has dimension 0. Furthermore an easy application of Bezout's theorem shows: whenever $\dim \Gamma = 0$ then Γ has 20 points counted with multiplicity.

Definition II.1.7. Suppose k is an algebraically closed field. Let X be a 0-dimensional k-scheme with structure sheaf $\mathcal{O}_X$. Let $p \in X$. Then the multiplicity of p is given by $m(p) = \dim_k \mathcal{O}_{X,p}$ where $\mathcal{O}_{X,p}$ is the local ring at p.

We say that A is *generic* if the point scheme Γ is finite. If in addition A is a quantum $\mathbb{P}^3$, we say A is a *generic quantum* $\mathbb{P}^3$.

Although AS-regular algebras can be thought of as deformations of the commutative polynomial ring, it is the generic ones which are the furthest from being polynomial. The point scheme of $k[x_1, x_2, x_3, x_4]$ is the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$. As another example, the Sklyanin algebras have 1-dimensional point schemes.

The point scheme has more structure. In the case of a quantum $\mathbb{P}^3$ it is the graph of an automorphism. Let $E = \pi_1(\Gamma)$ where $\pi_1 : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ is the projection onto the first factor. Here is the precise result.

Theorem II.1.8. *[21] Let $A = T(V^*)/I$ be an $\mathbb{N}$-graded connected k-algebra which satisfies the following.*

1) $H_A(t) = \dfrac{1}{(1-t)^4}$.

2) A is Noetherian and Auslander-regular of global dimension 4.

3) A is Cohen-Macaulay.

Let Γ denote the point scheme as above and $\pi_i : \mathbb{P}(V) \times \mathbb{P}(V) \to \mathbb{P}(V)$ be the projection on the ith factor. Set $E_i = \pi_i(\Gamma)$. Then $\pi_i : \Gamma \to E_i$ is an isomorphism of schemes for $i = 1, 2$, $E_1 = E_2$, and Γ is the graph of an automorphism $\tau : E \to E$.

We follow [13] for the definitions of Auslander-regular and Cohen-Macaulay. Let R be a ring and M a finitely generated R-module. The grade of M is given by

$$\mathrm{grade}(M) = \inf\{p \mid \mathrm{Ext}^p_R(M, R) \neq 0\}.$$

R is Auslander-regular of global dimension $n < \infty$ if: for every $p \geq 0$, every submodule N of $\mathrm{Ext}^p_R(M, R)$ has $\mathrm{grade}(N) \geq p$. R is Cohen-Macaulay if for every nonzero finitely generated module M,

$$\mathrm{GKdim}\, M + \mathrm{grade}(M) = \mathrm{GKdim}\, A.$$

An interesting fact about the point scheme is that it parametrizes the point modules which we define next.

Definition II.1.9. A point module P of A is a cyclic, graded, A-module which is generated in degree 0 and has Hilbert series $H_P(t) = \dfrac{1}{1-t}$.

Given $(p, \tau(p)) \in \Gamma$ or equivalently $p \in E$ we (following [18]) define a point module $M(p)$ as follows. Identify $\mathbb{P}^3$ with $\mathbb{P}(A_1)$ and choose a nondegenerate symmetric bilinear form on A_1. Then $p \in \mathbb{P}^3$ defines $p^\perp \subset A_1$. Let $M(p) = A/p^\perp A$, then P is a point module by [18]. Conversely given a point module P, we may define a point $(p, \tau(p)) \in \Gamma$ by the following. Let $p = (\mathrm{Ann}_{A_1} P)^\perp$ then $(p, \tau(p)) \in \Gamma$. It follows that the action of τ on E induces an action of τ on the point modules given by $\tau.P = P[-1]_{\geq 0}$.

II.2. Algebras as modules over their centers.

We now wish to prove some general results which will enable us to conclude that all of the examples we construct in III (and in fact all known examples) of generic quantum $\mathbb{P}^3$'s are finitely generated modules over their centers.

The following result is completely well known and standard but for lack of a good reference we include the statement and its proof. It is the analogue of the NAK lemma in the category of graded modules.

Lemma II.2.1. *Let R be an $\mathbb{N}$-graded commutative ring and let I be a homogeneous ideal generated by elements of strictly positive degree. Suppose M is a $\mathbb{Z}$-graded R-module with $IM = M$ and $M_n = 0$ for all $n << 0$. Then $M = 0$.*

Proof. Choose $j \in \mathbb{Z}$ with $M_i = 0$ for all $i < j$. Then $M_j = (IM)_j = 0$ where the second equality follows from the fact that I is generated in positive degree and $M_{<j} = 0$. By induction $M = 0$. □

Lemma II.2.2. *Let R be an $\mathbb{N}$-graded commutative ring and $q \in R_d$, $d > 0$. Suppose M is a $\mathbb{Z}$-graded, bounded below R-module with M/qM finitely generated as an R/qR-module. Then M is a finitely generated R-module.*

Proof. Let $\bar{m}_1, \ldots, \bar{m}_n$ be homogeneous generators of M/qM over R/qR. Choose homogeneous preimages $m_1, \ldots, m_n$ of the $\bar{m}_i$ in M and define $M' = \sum_i Rm_i \leq M$, and $N = M/M'$. Then $qN = \dfrac{qM + M'}{M'} = \dfrac{M}{M'} = N$. Then II.2.1 gives $N = 0$.

□

Proposition II.2.3. *Let k be a field, A an $\mathbb{N}$-graded k-algebra, and Z a graded central subalgebra. Let $q_1, \ldots, q_n \in Z$ be central homogeneous elements of positive degree with $A/(q_1, \ldots, q_n)A$ a finitely generated $Z/(q_1, \ldots, q_n)Z$-module. Then A is a finitely generated Z-module.*

Proof. Induct on n. For the case $n = 1$ let $M = A/qA$. Then M is a bounded below, finitely generated Z/qZ module so II.2.2 implies A is a finitely generated Z-module.

Now assume the result for $n \geq 1$, and suppose $q_1, \ldots, q_{n+1}$ are homogeneous central elements of positive degree in Z with $A/(q_1, \ldots, q_{n+1})A$ a finitely generated $Z/(q_1, \ldots, q_{n+1})Z$-module. By the $n = 1$ case $A/(q_1, \ldots, q_n)A$ is finitely generated over $Z/(q_1, \ldots, q_n)Z$ and by induction, A is finite over Z. □

The existence of a central regular sequence in an algebra A gives extremely powerful information about the structure of A as a module over its center $Z(A)$. We have the following theorem.

Theorem II.2.4. *Let A be a finitely generated $\mathbb{N}$-graded k-algebra with $H_A(t) = \dfrac{p(t)}{(1-t)^n}$, with $p(t) \in k[t]$ and $n \in \mathbb{N}$. Suppose that $a_1, \ldots, a_n$ is a regular sequence of homogeneous elements of positive degree which are all central. Let $A' = A/(a_1, \ldots, a_n)$ and let $Z(A)$ denote the center of A. Then:*

1) A is a finitely generated module over $Z(A)$.

2) $C := k[a_1, \ldots a_n]$ is a weighted polynomial ring.

3) A' is a finite-dimensional ring and A is a free C-module of rank $\dim_k A'$.

Proof. Since $C \subset Z(A)$, 1) follows from 3). We prove 3) first. Let $d_i = \deg a_i$ for $1 \leq i \leq n$. By II.1.5 we have

$$H_{A'}(t) = \prod_{i=1}^{n}(1 - t^{d_i})H_A(t).$$

From the hypothesis on the Hilbert series of A, it follows that $r = H_{A'}(1) < \infty$. So A' is a finite-dimensional k-algebra. Let $r = \dim_k A'$. By II.2.3, A is a finitely generated C-module. We will show A is free of rank r after we prove 2).

Since A is finitely generated over C we know that $\operatorname{GKdim} C = \operatorname{GKdim} A = n$. Let $R = k[x_1, \ldots x_n]$ be the weighted polynomial ring on n-variables where $\deg x_i = d_i$, $1 \leq i \leq n$. It is well known that R is GK-critical, [11]. Let $\pi : R \twoheadrightarrow C$ be the canonical projection map. Since $\operatorname{GKdim} C = \operatorname{GKdim} R = n$ and since R is GK-critical we must have $\ker \pi = 0$. Therefore $C \cong R$. This completes the proof of 2).

We now show A is free over C of rank r. Choose a homogeneous basis $v_1, \ldots, v_r$ for A' as a k-vector space. The proof of II.2.2 shows that if we choose homogeneous preimages of $v_1, \ldots v_r$ say $u_1, \ldots u_r$ in A then the u_i generate A over C. Let V be the k-span of $u_1, \ldots, u_r$ then

$$H_V(t) = H_{A'}(t) = \prod_{i=1}^{n}(1 - t^{d_i})H_A(t).$$

Now consider the multiplication map $m : V \otimes_k C \to A$. Then m is surjective and

$$H_{V \otimes_k C} = H_V \cdot H_C = \prod_{i=1}^{n}(1 - t^{d_i})H_A(t) \cdot \frac{1}{\prod_{i=1}^{n}(1 - t^{d_i})} = H_A.$$

Hence m is an isomorphism and so A is a free C-module of rank r. □

The following theorem, combined with II.2.4, will enable us to prove that the quantum $\mathbb{P}^3$'s, which we construct next, are all Noetherian domains. We include the statement of the theorem for convenience and because it is a beautiful result. It is due to Artin, Tate, and Van den Bergh, [4].

Theorem II.2.5. *[4] Let A be an AS-regular algebra of dimension $d \leq 4$. If A is Noetherian and* $\operatorname{GKdim} A = d$ *then A is a domain.*

CHAPTER III

EXAMPLES OF GENERIC QUANTUM $\mathbb{P}^3$'s.

III.1. The Shelton-Tingey algebra and its twists.

One of the main goals of this thesis is to study the properties exhibited by generic quantum $\mathbb{P}^3$'s. The first example of a generic regular algebra of dimension 4 which is not a Clifford algebra is the following example constructed by Shelton-Tingey [17].

Let i denote a square root of -1 in the algebraically closed field k. Let $[a_1, a_2, a_3, a_4]$ denote homogeneous coordinates on $\mathbb{P}^3$.

Theorem III.1.1. *[17] Let A be the k-algebra with generators x_1, x_2, x_3, x_4 and relations*

$$r_1 = x_3 \otimes x_1 - x_1 \otimes x_3 + x_2 \otimes x_2 \qquad r_2 = ix_4 \otimes x_1 + x_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 - x_2 \otimes x_4 + x_3 \otimes x_3 \qquad r_4 = ix_3 \otimes x_2 + x_2 \otimes x_3$$

$$r_5 = x_1 \otimes x_1 - x_3 \otimes x_3 \qquad r_6 = x_2 \otimes x_2 - x_4 \otimes x_4$$

Then A is a generic quantum $\mathbb{P}^3$ which is a Noetherian domain and whose point scheme Γ consists of 20 distinct points. The points are $e_1 = [1, 0, 0, 0]$, $e_2 = [0, 1, 0, 0]$,

$e_3 = [0, 0, 1, 0]$, $e_4 = [0, 0, 0, 1]$, *and 16 points of the form* $[1, a_2, a_3, a_4]$ *where:*

$$1 - 4a_4^4 + a_4^8 = 0$$

$$a_2^2 - a_4^2 - 4ia_2a_4^3 + ia_2a_4^7 = 0$$

$$a_3 = 8ia_4^2 - 15a_2a_4^3 + 2ia_4^6 - 4a_2a_4^7$$

The automorphism τ *on* E *is given by* $\tau(e_1) = e_2$, $\tau(e_2) = e_1$, $\tau(e_3) = e_4$, $\tau(e_4) = e_3$, *and* $\tau([1, a_2, a_3, a_4]) = [1, ia_2/a_3^2, 1/a_3, -ia_4]$.

For the rest of this section we will denote by A the above example.

One would like to understand A better by putting it into a larger family. One way of doing this is to compute its graded twists. Given an $\mathbb{N}$-graded algebra B and a graded automorphism σ of B, one forms the right graded twist $(B^\sigma, *)$ as follows. As graded vector spaces $B^\sigma = B$, but we twist the multiplication via:

$$a * b = ab^{\sigma^k}$$

for $a \in B_k, b \in B_l$. In [31], J. Zhang proves that twisting does not change the representation theory, that is the categories $\operatorname{Gr}\operatorname{Mod}(B^\sigma)$ and $\operatorname{Gr}\operatorname{Mod}(B)$ are equivalent. Thus the scheme E is a twisting invariant. However, as will be demonstrated by the examples, the orbit structure of the automorphism τ is *not* an invariant. That is, the point scheme Γ is *not* a twisting invariant.

We also note that the center of an algebra is not a twist invariant. Furthermore the property of being "finite over center" is not a twist invariant.

We wish to find the graded automorphisms of A so that we can examine the twists of A. There is an action of k^* on $\mathrm{Aut}_{\mathrm{Gr}}(A)$ by defining $(\lambda.\varphi)(a) = \varphi(\lambda a)$ for $\lambda \in k^*$, $\varphi \in \mathrm{Aut}_{\mathrm{Gr}}(A)$, and $a \in A$. Note that if $\lambda \in k^*$ and $\varphi \in \mathrm{Aut}_{\mathrm{Gr}}(A)$ then $A^{\varphi} \cong A^{\lambda\varphi}$. So to determine all twists we need only compute $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$. We note that k^* is a normal subgroup of $\mathrm{Aut}_{\mathrm{Gr}}(A)$ so that $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ is a group.

Definition III.1.2. Suppose $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)$ and let M be a right A-module. Define the right A-module σ^*M via:

- $\sigma^*M = M$ as sets
- $m.x = m\sigma(x)$ for $m \in \sigma^*M$ and $x \in A$.

Let P be a point module and $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)$. As σ is an automorphism, σ^*P is cyclic and generated in degree 0. Since σ is graded, the Hilbert series of σ^*P is $\dfrac{1}{1-t}$. So σ^*P is a point module.

We also have an action of $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ on $\mathbb{P}^3$ by identifying $\mathbb{P}(A_1)$ and $\mathbb{P}^3$. So by restriction, we have an action of $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ on E. If $M(p)$ is the point module corresponding to $p \in E$ then $\sigma^*M(p)$ corresponds to the point $\sigma^{-1}(p)$.

Lemma III.1.3. *Let $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ and τ the automorphism of E. Then the actions of σ and τ on the set of point modules commute.*

Proof. Let P be a point module. Then $\tau.\sigma^*(P) = (\sigma^*P)[-1] = \sigma^*(P[-1]) = \sigma^*(\tau.P)$.

□

Notice that the actions of σ and τ also commute on the set of closed points of E. We now have the following result.

Lemma III.1.4. *Let $(e_i, e_j) \in \Gamma$ be one of the points for A as given in III.1.1. Then $\sigma(e_i, e_j)$ is one of the pairs (e_1, e_2), (e_2, e_1), (e_3, e_4), (e_4, e_3).*

Proof. From III.1.3 we know that $\tau(\sigma(e_i, e_j)) = \sigma(\tau(e_i, e_j))$. If we apply τ again, we see that $\sigma((e_i, e_j))$ has order 2 with respect to τ. Since the pairs (e_1, e_2), (e_2, e_1), (e_3, e_4), (e_4, e_3) are all the elements of Γ of order 2 with respect to τ, the result follows. □

Proposition III.1.5. *Let A be the Shelton-Tingey example. Then* $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^* \cong \mathbb{Z}_8$ *with generator*

$$\rho = \begin{pmatrix} 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. Let us first write down the conditions for an element $r \in T(V^*)_2$ to be in the span of the defining relations $r_1, \ldots, r_6$. Consider the r_i as 4 x 4 matrices by identifying $\sum_{i,j} a_{ij} x_i \otimes x_j \leftrightarrow A = (a_{ij})$, $a_{ij} \in k$.

Let $r = ar_1 + br_2 + cr_3 + dr_4 + er_5 + fr_6$. Then

$$r = \begin{pmatrix} e & 0 & -a & b \\ 0 & a+f & d & -c \\ a & id & c-e & 0 \\ ib & c & 0 & -f \end{pmatrix}.$$

So a 4 x 4 matrix $A = (a_{ij})$ is in the span of the relations if and only if:

$$a_{12} = 0 \qquad a_{21} = 0 \qquad a_{34} = 0 \qquad a_{43} = 0 \qquad a_{13} = -a_{31}$$

$$a_{24} = -a_{42} \qquad a_{14} = -ia_{41} \qquad a_{23} = -ia_{32} \qquad a_{22} = a_{31} - a_{44} \qquad a_{33} = a_{42} - a_{11}$$

Suppose $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$, then σ is determined by an invertible linear map on A_1 modulo scalars, in other words by an element of $\mathbb{P}\mathrm{Gl}(A_1)$. By III.1.4 we may assume $\sigma(e_1, e_2) = (e_i, e_j)$ where $(i, j) \in \{(1, 2), (2, 1), (3, 4), (4, 3)\}$. We wish to rule out the cases $(i, j) = (3, 4)$ and $(i, j) = (4, 3)$.

Suppose that $(i, j) = (3, 4)$. There are two possibilities for $\sigma(e_3, e_4)$. Either (i) $\sigma(e_3, e_4) = (e_1, e_2)$ or (ii) $\sigma(e_3, e_4) = (e_2, e_1)$. Suppose (i) $\sigma(e_3, e_4) = (e_1, e_2)$ and consider the action of σ on the relation $f = ix_3x_2 + x_2x_3$. Modulo the action of k^*, we may assume $\sigma(x_3) = x_1$, and $\sigma(x_2) = ax_4$, for some $a \in k^*$. Then $\sigma(f) = a(ix_1x_4 + x_4x_1)$ which is not in the span of the relations. Suppose (ii) $\sigma(e_3, e_4) = (e_2, e_1)$. Then we may assume $\sigma(x_2) = x_4$ and $\sigma(x_3) = bx_2$ for some $b \in k^*$. Then $\sigma(f) = b(ix_2x_4 + x_4x_2)$ which is not in the span of the relations. Hence the case $(i, j) = (3, 4)$ is not possible.

An analogous argument shows that the case $(i, j) = (4, 3)$ is impossible.

We now wish to rule out the cases where one of (e_1, e_2) and (e_3, e_4) is fixed while the other pair is transposed.

Suppose that $\sigma(e_1, e_2) = (e_1, e_2)$ and $\sigma(e_3, e_4) = (e_4, e_3)$. We may assume that $\sigma(x_1) = x_1$, $\sigma(x_2) = ax_2$ and $\sigma(x_3) = bx_4$ for $a, b \in k^*$. Then $\sigma(x_3x_1 - x_1x_3 + x_2^2) = bx_4x_1 - bx_1x_4 + ax_2^2$ which is not in the span of the relations. A similar argument shows that the case $\sigma(e_1, e_2) = (e_2, e_1)$, $\sigma(e_3, e_4) = (e_3, e_4)$ is impossible.

We now show that the remaining cases

1. $\sigma(e_1, e_2) = (e_1, e_2)$, $\sigma(e_3, e_4) = (e_3, e_4)$, and

2. $\sigma(e_1, e_2) = (e_2, e_1)$, $\sigma(e_3, e_4) = (e_4, e_3)$

are possible.

Consider the diagonal case $\sigma(e_1, e_2) = (e_1, e_2)$, $\sigma(e_3, e_4) = (e_3, e_4)$. Assume $\sigma(x_4) = x_4$, $\sigma(x_1) = ax_1$, $\sigma(x_2) = bx_2$, $\sigma(x_3) = cx_3$ for some $a, b, c \in k^*$. Then applying σ to the relations we have

$$
\begin{array}{ll}
acx_3x_1 - acx_1x_3 + b^2x_2^2 & a(ix_4x_1 + x_1x_4) \\
bx_4x_2 - bx_2x_4 + c^2x_3^2 & bc(ix_3x_2 + x_2x_3) \\
a^2x_1^2 - c^2x_3^2 & b^2x_2^2 - x_4^2
\end{array}
$$

It immediately follows that $b^2 = 1$; hence $ac = 1$. So $a^2 = \pm 1$. Now consider the 4 possibilities for a.

$$a = 1 \Rightarrow c = 1 \Rightarrow b = 1$$

$$a = -1 \Rightarrow c = -1 \Rightarrow b = 1$$

$$a = i \Rightarrow c = -i \Rightarrow b = -1$$

$$a = -i \Rightarrow c = i \Rightarrow b = -1$$

This yields 4 diagonal matrices. The case $\sigma(e_1, e_2) = (e_2, e_1)$, $\sigma(e_3, e_4) = (e_4, e_3)$ is exactly analogous to the diagonal case and yields 4 automorphisms including the matrix

$$\rho = \begin{pmatrix} 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now ρ has order 8 and since we have exactly 8 elements in $\mathbb{P}\mathrm{Gl}(A_1)$ the result follows.

□

We record the 8 automorphisms in $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ in the following definition. An element of $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ is uniquely determined by an element of $\mathbb{P}\mathrm{Gl}(A_1)$. We fix the basis $\{x_1, x_2, x_3, x_4\}$ for A_1, and let $E_{i,j}$, $1 \leq i, j \leq n$ denote the elementary matrix units.

Definition III.1.6. The elements of $\mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ are given by the identity, I, and

$$d_1 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4} \qquad d_2 = -iE_{1,1} - E_{2,2} + iE_{3,3} + E_{4,4}$$

$$d_3 = iE_{1,1} - E_{2,2} - iE_{3,3} + E_{4,4} \qquad s_1 = E_{1,2} + E_{2,1} + E_{3,4} + E_{4,3}$$

$$s_2 = -E_{1,2} + E_{2,1} - E_{3,4} + E_{4,3} \qquad s_3 = iE_{1,2} - E_{2,1} - iE_{3,4} + E_{4,3}$$

$$s_4 = -iE_{1,2} - E_{2,1} + iE_{3,4} + E_{4,3}$$

We now write down the generators and relations for the twists of A for the fixed basis $\{x_1, x_2, x_3, x_4\}$ of A_1. Note that if

$$\sum_{i=1}^{4} \lambda_i \otimes x_i, \quad \lambda_i \in A_1$$

is a quadratic relation of A then

$$\sum_{i=1}^{4} \lambda_i^\sigma \otimes x_i$$

is a quadratic relation in A^σ.

The following give the relations for A^σ, $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$ for the fixed basis $\{x_1, x_2, x_3, x_4\}$ of A_1^σ.

Example III.1.7. A^{d_1} :

$$r_1 = -x_3 \otimes x_1 + x_1 \otimes x_3 + x_2 \otimes x_2 \qquad r_2 = ix_4 \otimes x_1 - x_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 - x_2 \otimes x_4 - x_3 \otimes x_3 \qquad r_4 = -ix_3 \otimes x_2 + x_2 \otimes x_3$$

$$r_5 = -x_1 \otimes x_1 + x_3 \otimes x_3 \qquad r_6 = x_2 \otimes x_2 - x_4 \otimes x_4$$

Example III.1.8. A^{d_2} :

$$r_1 = ix_3 \otimes x_1 + ix_1 \otimes x_3 - x_2 \otimes x_2 \qquad r_2 = ix_4 \otimes x_1 - ix_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 + x_2 \otimes x_4 + ix_3 \otimes x_3 \qquad r_4 = -x_3 \otimes x_2 - x_2 \otimes x_3$$

$$r_5 = -ix_1 \otimes x_1 - ix_3 \otimes x_3 \qquad r_6 = -x_2 \otimes x_2 - x_4 \otimes x_4$$

Example III.1.9. A^{d_3} :

$$r_1 = -ix_3 \otimes x_1 - ix_1 \otimes x_3 - x_2 \otimes x_2 \qquad r_2 = ix_4 \otimes x_1 + ix_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 + x_2 \otimes x_4 - ix_3 \otimes x_3 \qquad r_4 = x_3 \otimes x_2 - x_2 \otimes x_3$$

$$r_5 = ix_1 \otimes x_1 + ix_3 \otimes x_3 \qquad r_6 = -x_2 \otimes x_2 - x_4 \otimes x_4$$

Example III.1.10. A^{s_1} :

$$r_1 = x_4 \otimes x_1 - x_2 \otimes x_3 + x_1 \otimes x_2 \qquad r_2 = ix_3 \otimes x_1 + x_2 \otimes x_4$$

$$r_3 = x_3 \otimes x_2 - x_1 \otimes x_4 + x_4 \otimes x_3 \qquad r_4 = ix_4 \otimes x_2 + x_1 \otimes x_3$$

$$r_5 = x_2 \otimes x_1 - x_4 \otimes x_3 \qquad r_6 = x_1 \otimes x_2 - x_3 \otimes x_4$$

Example III.1.11. A^{s_2} :

$$r_1 = x_4 \otimes x_1 - x_2 \otimes x_3 - x_1 \otimes x_2 \qquad r_2 = -ix_3 \otimes x_1 + x_2 \otimes x_4$$

$$r_3 = -x_3 \otimes x_2 + x_1 \otimes x_4 + x_4 \otimes x_3 \qquad r_4 = ix_4 \otimes x_2 - x_1 \otimes x_3$$

$$r_5 = x_2 \otimes x_1 - x_4 \otimes x_3 \qquad r_6 = -x_1 \otimes x_2 + x_3 \otimes x_4$$

Example III.1.12. A^{s_3} :

$$r_1 = x_4 \otimes x_1 + x_2 \otimes x_3 + ix_1 \otimes x_2 \qquad r_2 = x_3 \otimes x_1 - x_2 \otimes x_4$$

$$r_3 = -ix_3 \otimes x_2 - ix_1 \otimes x_4 + x_4 \otimes x_3 \qquad r_4 = ix_4 \otimes x_2 - ix_1 \otimes x_3$$

$$r_5 = -x_2 \otimes x_1 - x_4 \otimes x_3 \qquad r_6 = ix_1 \otimes x_2 + ix_3 \otimes x_4$$

Example III.1.13. A^{s_4} :

$$r_1 = x_4 \otimes x_1 + x_2 \otimes x_3 - ix_1 \otimes x_2 \qquad r_2 = -x_3 \otimes x_1 - x_2 \otimes x_4$$

$$r_3 = ix_3 \otimes x_2 + ix_1 \otimes x_4 + x_4 \otimes x_3 \qquad r_4 = ix_4 \otimes x_2 - ix_1 \otimes x_3$$

$$r_5 = -x_2 \otimes x_1 - x_4 \otimes x_3 \qquad r_6 = -ix_1 \otimes x_2 - ix_3 \otimes x_4$$

We now prove that these 8 examples are not Clifford algebras. From this it immediately follows that the original example is not a twist of a Clifford algebra. For if A was a twist of a Clifford algebra say R, then R would be a twist of A. Recall that if R is a Clifford algebra over a field k with linear generators $x_1, \ldots, x_n$ then the center of R is the polynomial ring $k[x_1^2, \ldots, x_n^2]$, see [12].

Theorem III.1.14. *For the examples A^σ above, let Z^σ denote the center of A^σ.*

1) For A^σ, $\sigma \in \{I, d_1, s_1, s_2, s_3, s_4\}$, we have $Z_2^\sigma = 0$.

2) For A^σ, $\sigma \in \{d_2, d_3\}$, we have $Z_2^\sigma = \mathrm{Span}_k\{x_1^2, x_2^2\}$.

Proof. Since A^σ is generated by x_1, x_2, x_3, x_4, to compute the center in degree 2 one needs to see when a generic element in the 10-dimensional space A_2^σ commutes with the x_i's. This is routine but tedious. The program AFFINE, written by Schelter, is used to do the calculations. □

Corollary III.1.15. *Let A^σ be as above.*

1) A^σ is not a Clifford algebra for all $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)/k^*$.

2) The original algebra A is not a twist of a Clifford algebra.

Proof. As noted above, a Clifford algebra on 4 linear generators has center which is a polynomial ring on the squares of the generators. This is not the case for A^σ by III.1.14.

□

We next turn to results about the point schemes (E, τ) of these twists. As noted above, twisting does not change the points of E, but does change the automorphism τ of E.

Definition III.1.16. Let B be an $\mathbb{N}$-graded k-algebra and let $\sigma \in \mathrm{Aut}_{\mathrm{Gr}} B$. Let B^σ denote the right graded twist of B by σ. Let M be a graded right B-module. Define a graded right B^σ module M^σ via:

- $M^\sigma = M$ as graded vector spaces
- $m_i.x = m_i x^{\sigma^i}$ for $m_i \in M_i^\sigma$ and $x \in B^\sigma$.

Note that if P is a point module, then P^σ is also a point module.

Let $R = T(V^*)/I$ be a quantum $\mathbb{P}^3$ satisfying the hypotheses of II.1.8 and let $\sigma \in \operatorname{Aut}_{\mathrm{Gr}} R$. Let Γ denote the point scheme of R. Then Γ is the graph of an automorphism $\tau : E \to E$ for some $E \subset \mathbb{P}(V)$. Now we want to see what happens to Γ and τ when we twist by σ. We have:

Proposition III.1.17. *Let R, σ, Γ, E, and τ be as in the previous paragraph. Let Γ^σ denote the point scheme of R^σ. Then Γ^σ is the graph of $\sigma^{-1}\tau : E \to E$.*

Proof. Let $P = v_0 R$ be the point module for R corresponding to $(p, \tau(p)) \in \Gamma$. Let P^σ denote the twist of P as in III.1.16. Let $(p', q') \in \Gamma^\sigma$ denote the point corresponding to P^σ. Since twisting does not change the action in degree 0, we have $p = p'$. Choose $0 \neq v_1 \in P_1$, $0 \neq v_1^\sigma \in P_1^\sigma$. Then

$$\begin{aligned} q' &= (\operatorname{Ann}(v_1^\sigma)^\perp \\ &= \sigma^{-1}(\operatorname{Ann}(v_1))^\perp) \\ &= \sigma^{-1}\tau(p) \end{aligned}$$

This completes the proof. □

We now make some remarks about the computation of the point scheme Γ and the automorphism τ. Suppose we are given a quadratic relation

$$\sum_{i,j} c_{ij} x_i \otimes x_j, \quad c_{ij} \in k^*.$$

This defines a 2-form on $\mathbb{P}(V) \times \mathbb{P}(V)$. We denote coordinates on the respective factors of $\mathbb{P}(V) \times \mathbb{P}(V)$ as $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4]$. A satisfies the hypotheses of II.1.8 so the point scheme Γ is the graph of τ. The algebras A^σ have reduced point schemes since they have the maximal number of points, 20. Hence to compute τ it suffices to find formulas for the b_i's in terms of the a_i's. The following result is a straightforward computation.

Theorem III.1.18. *Let A^σ be one of the twists of A. Let τ' denote the automorphism giving the graph of Γ^σ.*

1) Let $\sigma = d_1$. Then τ' is given by $\tau'(e_1) = e_2$, $\tau'(e_2) = e_1$, $\tau'(e_3) = e_4$, $\tau'(e_4) = e_3$ and $\tau'([1, a_2, a_3, a_4]) = [1, \frac{-ia_2}{a_3^2}, \frac{1}{a_3}, ia_4]$.

2) Let $\sigma \in \{d_2, d_3\}$. Then τ' is given by $\tau'(e_1) = e_2$, $\tau'(e_2) = e_1$, $\tau'(e_3) = e_4$, $\tau'(e_4) = e_3$ and $\tau'([1, a_2, a_3, a_4]) = [1, \frac{a_2}{a_3^2}, \frac{-1}{a_3}, a_4]$.

3) Let $\sigma = s_1$. Then τ' is given by $\tau'(e_i) = e_i$, for $i = 1, 2, 3, 4$ and $\tau([1, a_2, a_3, a_4]) = [1, \frac{-ia_3^2}{a_2}, \frac{a_2}{a_4}, \frac{-ia_3}{a_2}]$.

4) Let $\sigma = s_2$. Then τ' is given by $\tau'(e_i) = e_i$, for $i = 1, 2, 3, 4$ and $\tau([1, a_2, a_3, a_4]) = [1, \frac{ia_3^2}{a_2}, \frac{a_2}{a_4}, \frac{ia_3}{a_2}]$.

5) Let $\sigma = s_3$. Then τ' is given by $\tau'(e_i) = e_i$, for $i = 1, 2, 3, 4$ and $\tau([1, a_2, a_3, a_4]) = [1, \frac{-a_3^2}{a_2}, \frac{-a_2}{a_4}, \frac{a_3}{a_2}]$.

6) Let $\sigma = s_4$. Then τ' is given by $\tau(e_i) = e_i$, for $i = 1, 2, 3, 4$ and $\tau([1, a_2, a_3, a_4]) = [1, \frac{a_3^2}{a_2}, \frac{-a_2}{a_4}, \frac{-a_3}{a_2}]$.

Note that for both of the twists A^{d_2}, A^{d_3} the orbit structure of τ' is given by 10 orbits of order 2, whereas for A^{s_4} the orbit structure is given by 4 orbits of order 1 and 4 orbits of order 4. This contrasts with the orbit structure of τ in the original example A and shows that twisting can change the orbit structure.

We now wish to prove that A and its twists are all finite modules over their centers. In order to apply II.2.4, we first need to find central regular sequences in these examples. We have the following.

Theorem III.1.19. *Let A denote the Shelton-Tingey example, then the twists of A have central regular sequences as follows:*

1) In A and A^{d_1} we have the central regular sequence $x_1^4, x_2^4,\ (x_1x_2)^4 + (x_2x_1)^4$, $(x_3x_4)^4 + (x_4x_3)^4$.

2) In A^{d_2} and A^{d_3} we have the central regular sequence $x_1^2, x_2^2,\ (x_1x_2)^2 + (x_2x_1)^2$, $(x_3x_4)^2 + (x_4x_3)^2$.

3) In A^{s_1} and A^{s_2} we have the central regular sequence $x_1^4 - x_2^4, x_4^4 - x_3^4,\ (x_1x_2)^2 + (x_2x_1)^2$, $x_2x_1x_2^2x_4^2 + i(x_2x_1)^2x_2x_3 + ix_1x_2x_2^2x_3^2 + i(x_1x_2)^2x_1x_4 + ix_1^2x_2^2x_2x_3 - x_1^2x_2x_1x_3^2 - ix_1^2x_2x_1x_2^2 + ix_1^2x_1x_2x_4^2 + ix_1^3x_2^2x_4$.

4) In A^{s_3} and A^{s_4} we have the central regular sequence $x_1^4 + x_2^4, x_3^4 + x_4^4,\ (x_1x_2)^2 + (x_2x_1)^2$, $x_2x_1x_2^2x_4^2 - i(x_2x_1)^2x_2x_3 + x_1x_2x_2^2x_3^2 + i(x_1x_2)^2x_1x_4 - ix_1^2x_2^2x_2x_3 + x_1^2x_2x_1x_3^2 + x_1^2x_2x_1x_2^2 + x_1^2x_1x_2x_4^2 + ix_1^2x_1x_2^2x_4$.

Proof. We will prove 2) for the algebra A^{d_3}. The other cases are similar and can be checked using the computer program AFFINE. Let's first recall the relations of A^{d_3}.

$$r_1 = -ix_3 \otimes x_1 - ix_1 \otimes x_3 - x_2 \otimes x_2 \qquad r_2 = ix_4 \otimes x_1 + ix_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 + x_2 \otimes x_4 - ix_3 \otimes x_3 \qquad r_4 = x_3 \otimes x_2 - x_2 \otimes x_3$$

$$r_5 = ix_1 \otimes x_1 + ix_3 \otimes x_3 \qquad r_6 = -x_2 \otimes x_2 - x_4 \otimes x_4$$

In this proof we will use B to denote A^{d_3}. Let $\Omega_1 = (x_1x_2)^2 + (x_2x_1)^2$ and $\Omega_2 = (x_3x_4)^2 + (x_4x_3)^2$. Let $\bar{s}$ be the sequence $x_1^2, x_2^2, \Omega_1, \Omega_2$. We will first prove that the elements $x_1^2, x_2^2, \Omega_1, \Omega_2$ are central in B. To prove an element $a \in B$ is central, it suffices to check that a commutes with the generators x_i, $1 \leq i \leq 4$. For x_1^2 we have

$$x_1^2x_2 = -x_3^2(x_2) = -x_2x_3^2 = x_2x_1^2$$

$$x_1^2x_3 = -x_3^2x_3 = x_3x_1^2$$

$$x_1^2x_4 = -x_1x_4x_1 = x_4x_1^2$$

Thus x_1^2 is central. The computation showing that x_2^2 is central is similar. For Ω_1 we have:

$$\Omega_1x_1 = (x_1x_2)^2x_1 + (x_2x_1)^2x_1 = x_1(x_2x_1)^2 + x_1(x_1x_2)^2$$

$$\Omega_1x_2 = (x_1x_2)^2x_2 + (x_2x_1)^2x_2 = x_2(x_2x_1)^2 + x_2(x_1x_2)^2$$

$$\Omega_1 x_3 = (x_1x_2)^2x_3 + (x_2x_1)^2x_3 = x_1x_2x_1x_3x_2 + x_2x_1x_2(-x_3x_1 + ix_2^2)$$
$$= x_1x_2(-x_3x_1 + ix_2^2)x_2 - x_2x_1x_3x_2x_1 + ix_2x_1x_2^3$$
$$= -x_1x_3x_2x_1x_2 + ix_1x_2^4 - x_2(-x_3x_1 + ix_2^2)x_2x_1 + ix_2x_1x_2^3$$
$$= (x_3x_1 - ix_2^2)x_2x_1x_2 + ix_1x_2^4 + x_3x_2x_1x_2x_1 - ix_2^4x_1 + ix_2x_1x_2^3$$
$$= x_3(x_1x_2)^2 + x_3(x_2x_1)^2$$

$$\Omega_1 x_4 = (x_1x_2)^2x_4 + (x_2x_1)^2x_4 = x_1x_2x_1(-x_4x_2 + ix_3^2) - x_2x_1x_2x_4x_1$$
$$= x_1x_2x_4x_1x_2 + ix_1x_2x_1x_3^2 - x_2x_1(-x_4x_2 + ix_3^2)x_1$$
$$= x_1(-x_4x_2 + ix_3^2)x_1x_2 + ix_1x_2x_1x_3^2 - x_2x_4x_1x_2^2x_1 - ix_2x_1x_3^2x_1$$
$$= x_4x_1x_2x_1x_2 + ix_1x_3^2x_1x_2 + ix_1x_2x_1x_3^2 - (-x_4x_2 + ix_3^2)x_1x_2x_1 - ix_2x_1x_3^2x_1$$
$$= x_4(x_1x_2)^2 + x_4(x_2x_1)^2$$

The proof that Ω_2 is central is similar to that of Ω_1.

We now show that $\bar{s}$ is regular. First define $B' = B/(x_1^2, x_2^2)$. Then B' has relations:

$$x_3x_1 = -x_1x_3 \qquad x_4x_1 = -x_1x_4 \qquad x_4x_2 = -x_2x_4 \qquad x_3x_2 = x_2x_3$$
$$x_1^2 = 0 \qquad x_2^2 = 0 \qquad x_3^2 = 0 \qquad x_4^2 = 0$$

Then B' is graded and we want to consider a monomial basis. Let $l(m)$ denote the length of a monomial m as a word in x_1, x_2, x_3, x_4. Let $j \in \mathbb{N}$ and define S_1^j to be the set of monomials in x_1, x_2 of length j. Similarly let S_2^j denote the set of monomials in x_3, x_4 of length j. Notice that since the squares of the generators are zero in

B', $S_1^j = \{x_1x_2\cdots, x_2x_1\cdots\}$ and $S_2^j = \{x_3x_4\cdots, x_4x_3\cdots\}$ The notation $x_ix_k\cdots$ is short for the string of length j obtained by alternating x_i and x_k. Let $S_i = \bigcup_{j\geq 0} S_i^j$ for $i = 1, 2$.

From the relations of B', we see that any monomial m in x_1, x_2, x_3, x_4 can be written uniquely as $m = m_1m_2$ where $m_1 \in S_1$ and $m_2 \in S_2$. We now compute $\dim_k B'_n$ for fixed $n \in \mathbb{N}$. A monomial basis for B'_n is given by

$$\mathcal{B}_n = \{m = m_1m_2 \mid l(m) = n, m_1 \in S_1, m_2 \in S_2\}.$$

Note $\mathcal{B}_0 = \{1\}$, and $\mathcal{B}_1 = \{x_1, x_2, x_3, x_4\}$. Suppose $n \geq 2$ and let $m = m_1m_2 \in \mathcal{B}_n$. There are two choices for $m_1 \in S_1$ with $1 \leq l(m_1) \leq n-1$ and for a fixed choice of m_1 there are two possibilities for $m_2 \in S_2$. That is, fixing the length of m_1, we get the 4 monomials $x_1x_2\cdots x_3x_4\cdots$, $x_1x_2\cdots x_4x_3\cdots$, $x_2x_1\cdots x_3x_4\cdots$, and $x_2x_1\cdots x_4x_3\cdots$. If $l(m_1) \in \{0, n\}$ we get the 4 monomials $x_3x_4\cdots$, $x_4x_3\cdots$, $x_1x_2\cdots$, and $x_2x_1\cdots$. Hence $\mathcal{B}_n$ contains exactly $4n$ monomials for $n \geq 1$. Therefore $H_{B'}(t) = 1 + \sum_{n\geq 1}(4n)t^n = \dfrac{(1-t^2)^2}{(1-t^4)}$. By II.1.5, (x_1^2, x_2^2) is a regular sequence.

We now show that Ω_1 is regular in B'. Since we've shown Ω_1 is central, we need only check that Ω_1 is left regular. Also since B' is graded it suffices to show: $\Omega_1 f = 0 \Rightarrow f = 0$ for f homogeneous.

Let $f \in B'_n$, $n \geq 1$. We order $\mathcal{B}_n$ as

$$\{b_1, \ldots, b_{2n-1}, b_{2n}, \ldots b_{4n-2}, b_{4n-1}, b_{4n}\}$$

where $b_1, \ldots, b_{2n-1}$ begin in x_1, $b_{2n}, \ldots, b_{4n-2}$ begin in x_2, and b_{4n-1}, b_{4n} begin in x_3, x_4

respectively. Write f in terms of the basis $\mathcal{B}_n$ as

$$f = \sum_{i=1}^{2n-1} c_i b_i + \sum_{i=2n}^{4n-2} c_i b_i + \sum_{i=4n-1}^{4n} c_i b_i,$$

for some $c_1 \dots c_{4n} \in k$. Suppose $\Omega_1 f = 0$. Then

$$\begin{aligned}\Omega_1 f &= ((x_1x_2)^2 + (x_2x_1)^2)(\sum_{i=1}^{2n-1} c_i b_i + \sum_{i=2n}^{4n-2} c_i b_i + \sum_{i=4n-1}^{4n} c_i b_i) \\ &= \sum_{i=1}^{2n-1} c_i (x_1x_2)^2 b_i + \sum_{i=2n}^{4n-2} c_i (x_2x_1)^2 b_i + (\sum_{i=4n-1}^{4n} c_i \Omega_1 b_i) \\ &= 0\end{aligned}$$

The last 2 lines give an equation in B'_{n+2} which is expressed in terms of the basis $\mathcal{B}_{n+2}$. Hence $c_1 \dots, c_{4n}$ are all zero and we get $f = 0$ as required.

To prove Ω_2 is regular modulo (x_1^2, x_2^2, Ω_1), we define $B'' = B'/\Omega_1$. In B'' we have the relation $(x_1x_2)^2 = -(x_2x_1)^2$. We can find a canonical basis of monomials for B''_n, where a monomial has the canonical form $(x_1x_2\cdots)^l m'$ with $l \in \mathbb{N}$ and m' a monomial in x_3, x_4. Since we have proved Ω_2 is central, it suffices to prove Ω_2 is right regular. This is done with an analogous argument as above.

So $\bar{s}$ is a regular sequence of central elements, and this completes the proof.

□

We now prove that the Shelton-Tingey example A and all of its right graded twists are finite modules over their centers.

Theorem III.1.20. *Let A denote the Shelton-Tingey example and let σ be one of* *the 8 automorphisms of A as given in III.1.5. Then A^σ is a finite module over its center. More precisely we have:*

1) The algebras A and A^{d_1} are free modules over the central subalgebras $C = k[x_1^4, x_2^4, (x_1x_2)^4 + (x_2x_1)^4, (x_3x_4)^4 + (x_4x_3)^4]$ of rank 1024.

2) The algebras A^{d_2} and A^{d_3} are free modules over the central subalgebras $C = k[x_1^2, x_2^2, (x_1x_2)^2 + (x_2x_1)^2, (x_3x_4)^2 + (x_4x_3)^2]$ of rank 64.

3) The algebras A^{s_1} and A^{s_2} are free modules over the central subalgebras $C = k[x_1^4 - x_2^4, x_4^4 - x_3^4, (x_1x_2)^2 + (x_2x_1)^2, x_2x_1x_2^2x_4^2 + i(x_2x_1)^2x_2x_3 + ix_1x_2x_2^2x_3^2 + i(x_1x_2)^2x_1x_4 + ix_1^2x_2^2x_2x_3 - x_1^2x_2x_1x_3^2 - ix_1^2x_2x_1x_2^2 + ix_1^2x_1x_2x_4^2 + ix_1^3x_2^2x_4]$ of rank 384.

4) The algebras A^{s_3} and A^{s_4} are free modules over the central subalgebras $C = k[x_1^4 + x_2^4, x_3^4 + x_4^4, (x_1x_2)^2 + (x_2x_1)^2, x_2x_1x_2^2x_4^2 - i(x_2x_1)^2x_2x_3 + x_1x_2x_2^2x_3^2 + i(x_1x_2)^2x_1x_4 - ix_1^2x_2^2x_2x_3 + x_1^2x_2x_1x_3^2 + x_1^2x_2x_1x_2^2 + x_1^2x_1x_2x_4^2 + ix_1^2x_1x_2^2x_4]$ of rank 384.

Proof. In each of 1), 2), 3), and 4), the generators of C form central regular sequences by III.1.19. Then II.2.4 implies that A^σ is finitely generated over its center, and that A^σ is free over C in each case. Finally, the actual rank is determined by the following. Suppose q_1, q_2, q_3, q_4 are the generators of C with degrees d_1, d_2, d_3, d_4 respectively. Let $p(t) = (\prod_{i=1}^{4}(1-t^{d_i}))H_A(t) = \frac{\prod_{i=1}^4(1-t^{d_i})}{(1-t)^4}$ which is a polynomial in t. Then the rank is given by $p(1)$. For example in 2. we have $p(t) = \frac{(1-t^2)^2(1-t^4)^2}{(1-t)^4} = (1+t)^4(1+t^2)^2$ and $p(1) = 64$. The other computations are analogous. □

III.2. Families of generic quantum $\mathbb{P}^3$'s.

To construct other generic examples we first review the notion of Ore extensions. Let R be a ring, $\theta \in \text{Aut}(R)$ and δ a right θ derivation, that is $\delta : R \to R$ is R-linear with

$$\delta(rs) = \delta(r)\theta(s) + r\delta(s), \text{ for } r, s \in R.$$

The right Ore extension with respect to this data is the ring $R[z; \theta, \delta]$ whose elements are polynomials $\sum_k z^k r_k$ in the indeterminate z with right coefficients $r_k \in R$ subject to the relations

$$rz = z\theta(r) + \delta(r) \text{ for } r \in R.$$

The usefulness of this construction for our purposes is as follows. Let R be a quantum $\mathbb{P}^2$ and consider the Ore extensions of R by a degree one indeterminate z via a graded automorphism and a degree 2 graded θ-derivation δ. These extensions are all quantum $\mathbb{P}^3$'s [6]. Given that the quantum $\mathbb{P}^2$'s are classified, we can start with an algebra R which has many quadratic normal elements (e.g. a skew polynomial ring), form the Ore extension, find a quadratic normal regular sequence q_1, q_2, q_3, q_4 and then construct the Koszul dual $(R/(q_1, q_2, q_3, q_4))^!$. This construction yields many new examples of generic $q\mathbb{P}^3$'s. This is what we shall discuss next.

Construction 1

In this subsection we will construct some examples of generic quantum $\mathbb{P}^3$'s from the first homogenized Weyl algebra, $A_1(k)$. We define this first.

Definition III.2.1. The first homogenized Weyl algebra $A_1(k)$ is the $\mathbb{N}$-graded k-algebra presented as

$$A_1(k) = \frac{k\langle x, y, z\rangle}{\langle [x,z], [y,z], [x,y] - z^2\rangle}$$

where $\deg(x) = \deg(y) = \deg(z) = 1$ and $[a,b] = ab - ba$.

Recall that $A_1(k)$ is a quantum $\mathbb{P}^2$. We now define an automorphism of $A_1(k)$ on $A_1(k)_1$ by

$$\theta_\lambda = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\lambda \in k$. To prove that θ_λ defines an automorphism of $A_1(k)$ we need to check that θ_λ preserves the span of the relations of $A_1(k)$. We have $\theta_\lambda([x,z]) = [x,z]$, $\theta_\lambda([y,z]) = \lambda[x,z] + [y,z]$, and $\theta_\lambda([x,y] - z^2) = [x,y] - z^2$.

Definition III.2.2. Let $R = A_1(k)[w;\theta]$ be the right Ore extension of $A_1(k)$ by a degree 1 indeterminate w and the automorphism θ_λ. Then R has a presentation as

$$\frac{k\langle x, y, z, w\rangle}{\langle [x,z], [y,z], [x,y] - z^2, [x,w], yw - \lambda wx - wy, [z,w]\rangle}.$$

Lemma III.2.3. *Let $\alpha \in k^*$. Define the sequence $s = (z^2, xw, x^2 + y^2, w^2 - \alpha xy + yz)$ of quadratic elements in R. Then s is a normal sequence.*

Proof. It is clear from the relations of R that z^2 is normal, in fact z is central. Now consider xw modulo z^2. It is clear that xw commutes with x, z, and w in R, so we need only consider $(xw)y$ modulo z^2. Computing modulo z^2, we have

$$\begin{aligned}(xw)y &= x(yw - \lambda wx)\\ &= (yx + z^2)w - \lambda x(xw)\\ &= (y - \lambda x)(xw)\end{aligned}$$

So xw is normal mod z^2.

Now notice that modulo (z^2, xw), all of the degree 1 elements commute. So x^2+y^2 and $w^2 - \alpha xy + yz$ are central, hence normal modulo (z^2, xw). This completes the proof. □

Lemma III.2.4. *Let s be the sequence from III.2.3. Then s is a regular sequence.*

Proof. Let $R' = R/(z^2, xw, x^2 + y^2)$. First notice that R' is a commutative ring. We now compute the Hilbert series of R'.

Let $\mathcal{B}_n$ for $n \geq 0$ denote a monomial basis for R'_n. Then it is not hard to see that $\mathcal{B}_0 = \{1\}$, $\mathcal{B}_1 = \{x, y, z, w\}$, $\mathcal{B}_2 = \{x^2, xy, xz, yz, yw, zw, w^2\}$ and $\mathcal{B}_n = \{x^n, x^{n-1}y, x^{n-1}z, x^{n-2}yz, yw^{n-1}, zw^{n-1}, yzw^{n-2}, w^n\}$. Therefore we have

$$H_{R'}(t) = 1 + 4t + 7t^2 + \sum_{n\geq 3} 8t^n = \frac{(1-t^2)^3}{(1-t)^4}.$$

By II.1.5, $(z^2, xw, x^2 + y^2)$ is a normal regular sequence.

It remains to show that $q_4 := w^2 - \alpha xy + yz$ is regular in R'. Since R' is commutative and q_4 is homogeneous, it suffices to show that q_4 is left regular on R'_n for all $n \geq 0$. Given the monomial basis $\mathcal{B} = \bigcup_{n\geq 0} \mathcal{B}_n$ for R', this is elementary linear algebra. This completes the proof. □

By III.2.3 and III.2.4 we know that s is a normal regular sequence in R. We now have the following theorem.

Theorem III.2.5. *Let S_α, $\alpha \in k^*$ be the $\mathbb{N}$-graded k-algebra on 4 degree 1 generators x_1, x_2, x_3, x_4 with relations*

$$r_1 = x_1 \otimes x_3 + x_3 \otimes x_1 \qquad r_2 = x_2 \otimes x_3 + x_3 \otimes x_2 - x_4 \otimes x_4$$

$$r_3 = x_1 \otimes x_2 + x_2 \otimes x_1 + \alpha x_4 \otimes x_4 \qquad r_4 = x_2 \otimes x_4 + x_4 \otimes x_2$$

$$r_5 = x_3 \otimes x_4 + x_4 \otimes x_3 \qquad r_6 = x_1 \otimes x_1 - x_2 \otimes x_2$$

Then S_α is a quantum $\mathbb{P}^3$ whose point scheme Γ consists of 5 distinct points given by $p_1 = [1, i, 0, 0], p_2 = [1, -i, 0, 0], e_1, e_3, e_4$ where $e_i = [0, \ldots, 1, \ldots]$ with 1 in the ith spot. The automorphism τ on these points is given by $p_1 \mapsto p_2$, $p_2 \mapsto p_1$, $e_1 \mapsto e_4$, $e_3 \mapsto e_3$ and $e_4 \mapsto e_1$.

We note that the change of basis; $x_1 \mapsto x_1, x_2 \mapsto x_2, x_3 \mapsto \alpha^{-1} x_3, x_4 \mapsto (\sqrt{\alpha})^{-1} x_4$ shows that $S_\alpha \cong S_1$. So that S_α does not depend on α. We set $S := S_1$.

Proof. Consider the algebra R given in III.2.2 and the normal regular sequence s given in III.2.3. We change notation via $x_1 = x, x_2 = y, x_3 = z, x_4 = w$. Computing the orthogonal relations to $[x_2, x_3], [x_1, x_2], -x_4 x_1, [x_2, x_4], [x_3, x_4], x_3^2, x_1 x_4, x_1^2 + x_2^2, x_4^2 - x_1 x_2 + x_2 x_3$, we get the relations $r_1, \ldots, r_6$. By II.1.6, S is a quantum $\mathbb{P}^3$.

We now compute the point scheme. We have to find the zeroes of the relations $r_1, \ldots, r_6$ in $\mathbb{P}^3 \times \mathbb{P}^3$. We coordinatize $\mathbb{P}^3 \times \mathbb{P}^3$ by using coordinates $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4]$ on the first and second factors of $\mathbb{P}^3 \times \mathbb{P}^3$ respectively. We consider three cases based on the decomposition $\mathbb{P}^3 \times \mathbb{P}^3 = (\mathbb{A}^3 \times \mathbb{A}^3) \cup (\mathbb{P}^2 \times \mathbb{P}^3) \cup (\mathbb{P}^3 \times \mathbb{P}^2)$ where $\mathbb{A}^3 \times \mathbb{A}^3$ is the affine subvariety given by $a_1 \neq 0, b_1 \neq 0$, $\mathbb{P}^2 \times \mathbb{P}^3$ is the subvariety given by $a_1 = 0$ and $\mathbb{P}^3 \times \mathbb{P}^2$ is the subvariety given by $b_1 = 0$.

Case 1: $a_1 = b_1 = 1$.

We must solve the following equations:

$$b_3 + a_3 = 0 \qquad a_2b_3 + a_3b_2 - a_4b_4 = 0 \qquad b_2 + a_2 + a_4b_4 = 0$$

$$a_2b_4 + a_4b_2 = 0 \qquad a_3b_4 + a_4b_3 = 0 \qquad 1 - a_2b_2 = 0$$

We have $b_3 = -a_3$ and $b_2 = a_2^{-1}$. Making these substitutions we get

$$-a_2a_3 + a_3a_2^{-1} - a_4b_4 = 0 \qquad a_2^{-1} + a_2 + a_4b_4 = 0$$

$$a_2b_4 + a_4a_2^{-1} = 0 \qquad a_3(b_4 - a_4) = 0$$

We can consider two cases: (a) $a_3 = 0$ or (b) $b_4 = a_4$. If $a_3 = 0$ then $a_4b_4 = 0$ so $a_2^{-1} + a_2 = 0$, i.e. $a_2 = \pm i$. Also $a_4 = b_4 = 0$. This gives the points $[1, \pm i, 0, 0]$.

If $a_4 = b_4$ and $a_3 \neq 0$ then $a_4(a_2 + a_2^{-1}) = 0$. If $a_4 = 0$ then as in the previous paragraph $a_2^2 = -1$. But then we have $a_3(a_2^{-1} - a_2) = 0$ so $a_3 = 0$, contradiction. Otherwise $a_2 + a_2^{-1} = 0$ so $a_4b_4 = 0$. From $a_2b_4 + a_4a_2^{-1} = 0$ we have $a_4 = 0$ which implies $a_3 = 0$, a contradiction. This concludes Case 1.

Case 2: $a_1 = 0$.

In this case we must solve the equations:

$$a_3b_1 = 0 \qquad a_2b_3 + a_3b_2 - a_4b_4 = 0 \qquad a_2b_1 + a_4b_4 = 0$$

$$a_2b_4 + a_4b_2 = 0 \qquad a_3b_4 + a_4b_3 = 0 \qquad -a_2b_2 = 0$$

There are 4 possibilities to look at:

(a) If $a_3 = a_2 = 0$ then we may assume $a_4 \neq 0$. So $b_4 = b_2 = b_3 = 0$ and we have the solution (e_4, e_1).

(b) If $a_3 = b_2 = 0$ then we have $a_2b_3 - a_4b_4 = 0$, $a_2b_1 + a_4b_4 = 0$, $a_2b_4 = 0$, and $a_4b_3 = 0$. If $a_2 = 0$ then we may assume $a_4 \neq 0$ and so $b_4 = b_3 = 0$. This gives the solution (e_4, e_1). If $b_4 = 0$ and $a_2 \neq 0$ then $b_3 = 0$. But then $b_1 \neq 0$ but this contradicts $a_2b_1 = 0$.

(c) If $b_1 = a_2 = 0$ then we have the equations $a_3b_2 - a_4b_4 = 0$, $a_4b_4 = 0$, $a_4b_2 = 0$, and $a_3b_4 + a_4b_3 = 0$. If $a_4 = 0$ then we may assume $a_3 \neq 0$. Then we have $b_2 = b_4 = 0$ which gives the solution (e_3, e_3). If $b_2 = 0$ and $a_4 \neq 0$ then $b_4 = b_3 = 0$ and there are no solutions.

(d) If $b_1 = b_2 = 0$ then we have $a_2b_3 - a_4b_4 = 0$, $a_4b_4 = 0$, $a_2b_4 = 0$, and $a_3b_4 + a_4b_3 = 0$. If $b_4 = 0$ then $b_3 \neq 0$ so $a_2 = a_4 = 0$ which gives the solution (e_3, e_3). Otherwise $b_4 \neq 0$ and then $a_2 = a_4 = a_3 = 0$ which is not a projective solution. This concludes Case2.

Case 3: $b_1 = 0$.

By symmetry in the relations $r_1, \ldots, r_6$, this case gives the solutions (e_1, e_4) and (e_3, e_3).

Therefore Γ consists of the 5 distinct points p_1, p_2, e_1, e_3, e_4 and it is clear from the above analysis that the action of τ is as stated.

$\square$

We wish to analyse the algebra S further. In particular we will prove that it is a Noetherian domain.

Lemma III.2.6. *Define the sequence of quadratic regular elements in S by $s = (x_1x_4 + x_4x_1, x_3^2, x_1x_2 + x_2x_1, x_1^2)$. Then s is a central regular sequence.*

Proof. It is straightforward to check that the elements in s are in fact all central in S. To prove the sequence is regular we note that S/s is the alternating algebra on 4 linear generators over k so its Hilbert series is $(1+t)^4$. We are done by II.1.5. □

Theorem III.2.7. *Let S be as in III.2.5 and $s = (x_1x_4 + x_4x_1, x_3^2, x_1x_2 + x_2x_1, x_1^2)$. Let $C = k[x_1x_4 + x_4x_1, x_3^2, x_1x_2 + x_2x_1, x_1^2]$ and Z denote the center of S. Then:*

1) S is a finitely generated module over Z.

2) C is a weighted polynomial ring.

3) S is a free module of rank 16 over C.

Proof. By III.2.6 we know that s is a central regular sequence and $H_{S/s}(t) = (1+t)^4$ so that $\dim_k S/s = 2^4 = 16$. The rest follows from II.2.4. □

Corollary III.2.8. *The algebra S is a Noetherian domain.*

Proof. From III.2.7 we know that S is a finitely generated free module over the polynomial subalgebra C. Since C is Noetherian we have that S is Noetherian. So by II.2.5 we get that S is a domain. □

This concludes the subsection Construction 1.

Construction 2

In this subsection we will construct more examples of generic quantum $\mathbb{P}^3$'s. We start with a skew polynomial ring of global dimension 3 and then form an Ore extension. This differs from construction 1 in that we use a nontrivial derivation for the extension. We then find various normal regular sequences and use II.1.6 to build families of generic quantum $\mathbb{P}^3$'s.

Consider the one parameter family of quantum $\mathbb{P}^2$'s given by $k\langle x_1, x_2, x_4\rangle$ with quadratic relations:

$$r_1 = x_1 \otimes x_2 - x_2 \otimes x_1 \qquad r_2 = x_1 \otimes x_4 - \zeta x_4 \otimes x_1 \qquad r_3 = x_2 \otimes x_4 - \zeta x_4 \otimes x_2$$

where $\zeta \in k^*$. Denote a member of this family by R_ζ. Define the automorphism $\theta \in \mathrm{Aut}(R_\zeta)$ given in degree 1 on the basis $\{x_1, x_2, x_4\}$ as

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix}.$$

Define the θ-derivation δ via the formula

$$\delta = (ax^2 + bxy + ay^2)(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$$

with $a, b \in k$. This gives a linear action of δ on R_1 and we extend δ to R by the Leibniz rule. Let $S = S_{a,b,\zeta} = R_{a,b,\zeta}[x_3; \theta, \delta]$ be the associated right Ore extension by the degree one indeterminate x_3.

Then S is generated by x_1, x_2, x_3, x_4 with relations

$$r_1 = x_1 \otimes x_2 - x_2 \otimes x_1$$

$$r_2 = x_1 \otimes x_4 - \zeta x_4 \otimes x_1$$

$$r_3 = x_2 \otimes x_4 - \zeta x_4 \otimes x_2$$

$$r_4 = x_1 \otimes x_3 - x_3 \otimes x_2 - ax_1 \otimes x_1 - bx_1 \otimes x_2 - ax_2 \otimes x_2$$

$$r_5 = x_1 \otimes x_3 - x_3 \otimes x_1 + x_2 \otimes x_3 - x_3 \otimes x_2$$

$$r_6 = x_3 \otimes x_4 - \zeta x_4 \otimes x_3.$$

We constructed the derivation δ so that the generator x_3 isn't normal in S but x_3^2 is normal.

Proposition III.2.9. *In the algebra S let $q_1 = x_3^2$ and $q_2 = x_2x_3 - x_3x_2$. Then q_1, q_2 is a normal regular sequence.*

Proof. By construction, q_1 is normal. S is Noetherian being an Ore extension of a Noetherian algebra so by [4][Theorem 3.9], S is a domain. So q_1 is regular. It is straightforward by direct computation to verify that q_2 is normal and regular modulo q_1. □

In the algebra $S/(q_1, q_2)$, the monomials in degree 2 are all normal. Recall that a normal element defines a graded homomorphism. In fact,

Lemma III.2.10. *In $S/(q_1, q_2)$,* *1) The monomials* $x_1^2, x_1x_2, x_2^2, x_1x_3$ *define the automorphisms* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix}$.

2) The monomials x_1x_4, x_2x_4, x_3x_4 *define the automorphisms* $\begin{pmatrix} \zeta^{-1} & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 0 & 0 \\ 0 & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}$.

3) The monomial x_4^2 *defines the automorphism* $\begin{pmatrix} \zeta^{-2} & 0 & 0 & 0 \\ 0 & \zeta^{-2} & 0 & 0 \\ 0 & 0 & \zeta^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Proof. This is a completely straightforward computation. For an element Ω, we write Ωx_i in the form $\varphi(x_i)\Omega$ for a linear map $\varphi : S_1 \to S_1$. □

Notice that according to the above lemma, when $\zeta = \pm 1$ many of the monomials define the same automorphism. Hence linear combinations of these monomials yield normal elements. Let S^+, S^- denote $S_{a,b,1}$, $S_{a,b,-1}$ respectively. The following examples were found by building elements q_3, q_4 for which the sequence q_1, q_2, q_3, q_4 is normal and regular in S and then finding the Koszul dual $S^!$. For each of the examples below, we have fixed the elements $q_1 = x_3^2$, $q_2 = x_2x_3 - x_3x_2$. We write down the quadratic elements q_3, q_4 for which q_1, q_2, q_3, q_4 give a normal regular sequence. Then we use II.1.6 to build the example. In the following 2 examples we set $c := 1/a$.

Example III.2.11. *Let $A = k\langle x_1, x_2, x_3, x_4\rangle$ with relations:*

$$r_1 = x_3 \otimes x_4 + x_4 \otimes x_3 + ix_1 \otimes x_2 + ix_2 \otimes x_1 - ibcx_2 \otimes x_2$$

$$r_2 = x_1 \otimes x_4 + x_4 \otimes x_1$$

$$r_3 = x_2 \otimes x_4 + x_4 \otimes x_2$$

$$r_4 = x_1 \otimes x_3 + x_3 \otimes x_1 + cx_2 \otimes x_2 + x_4 \otimes x_4$$

$$r_5 = x_2 \otimes x_3 + x_3 \otimes x_2 - cx_2 \otimes x_2$$

$$r_6 = x_1 \otimes x_1 - x_2 \otimes x_2.$$

where $bc \notin \{\pm 2\}$. Then:

1) A is a quantum $\mathbb{P}^3$ which is constructed from S^+ by the normal regular sequence $q_1, q_2, q_3 = x_1x_2 - ix_3x_4, q_4 = x_1x_3 - x_4^2$.

2) The point scheme Γ consists of 5 distinct points, $\{p_1, \ldots p_5\}$ and the automorphism τ has 2 orbits of order 2 and fixes p_5. The points are given by: $p_i = ([1, a_2, a_3, 0], [1, a_2^{-1}, -a_3a_2^{-1}, 0])$ for $i = 1, 2$ where a_2 satisfies $x^2 - bcx + 1$ and $a_3 = \dfrac{-a_2c}{a_2 - 1}$. The points p_3, p_4 are given by $([1, 1, a_3, a_4], [1, 1, c - a_3, -a_4])$ where $a_4^2 = -2c$ and $2i - 2a_3a_4 + a_4c - ibc = 0$. Finally, $p_5 = ([0, 0, 1, 0], [0, 0, 1, 0])$.

3) A is finite over its center as $x_1^2, x_3^2, x_1x_2 + x_2x_1, x_1x_3 + x_3x_1$ is a regular sequence of central elements in A.

4) A is a Noetherian domain.

Example III.2.12. *Let $A = k\langle x_1, x_2, x_3, x_4\rangle$ with relations:*

$$r_1 = x_1 \otimes x_2 + x_2 \otimes x_1 - bcx_2 \otimes x_2$$

$$r_2 = x_1 \otimes x_4 - x_4 \otimes x_1 + x_2 \otimes x_4 - x_4 \otimes x_2$$

$$r_3 = x_1 \otimes x_3 + x_3 \otimes x_1 + cx_2 \otimes x_2$$

$$r_4 = x_2 \otimes x_3 + x_3 \otimes x_2 - cx_2 \otimes x_2$$

$$r_5 = x_1 \otimes x_1 + x_4 \otimes x_4 - x_2 \otimes x_2$$

$$r_6 = x_3 \otimes x_4 - x_4 \otimes x_3.$$

where $bc \notin \{\pm 2, -1\}$ and $c \neq 0$. Then:

1) A is a quantum $\mathbb{P}^3$ which is constructed from S^- by the normal regular sequence $q_1, q_2, q_3 = x_1x_4 - x_2x_4, q_4 = x_4^2 - x_1^2$.

2) The point scheme Γ consists of 7 distinct points, $\{p_1, \dots p_7\}$ and the automorphism τ has 3 orbits of order 2 and fixes p_7. The points are given by: $p_1 = ([1,0,0,i],[1,0,0,i])$, $p_2 = ([1,0,0,-i],[1,0,0,-i])$. The points p_3, p_4 are given by $([1,a_2,a_3,0],[1,a_2^{-1},\frac{-a_3}{a_2},0])$, where a_2 satisfies $x^2 - bcx + 1 = 0$ and $a_3 = \frac{-a_2c}{a_2 - 1}$. And finally, $p_5 = ([1,-1,0,0],[0,0,0,1])$, $p_6 = ([0,0,0,1],[1,-1,0,0])$ and $p_7 = ([0,0,1,0],[0,0,1,0])$.

3) A is finite over its center as $x_1^2, x_1x_2 + x_2x_1, x_3^2, (x_4x_1)^2 + (x_1x_4)^2$ is a regular sequence of central elements in A.

4) A is a Noetherian domain.

Statements 1), 2), 3), 4) in Examples III.2.11, III.2.12 are proved as follows. For statement 1) we need only check that the sequence q_1, q_2, q_3, q_4 is a normal regular sequence. The normality of q_1, q_2, q_3, q_4 follows from III.2.9 and III.2.10. To prove regularity, we show that the Hilbert series of $S' = S/(q_1, q_2, q_3, q_4)$ is $(1+t)^4$. Finally II.1.6 implies A is a quantum $\mathbb{P}^3$.

Statement 2) is a computation of the zeroes of the relations in $\mathbb{P}^3 \times \mathbb{P}^3$ along the same lines as in the proof of III.2.5. For statement 3), we need to check that the given elements c_1, c_2, c_3, c_4 are central. It suffices to check that c_i commutes with the generators x_j, $1 \leq j \leq 4$. Then we compute the Hilbert series of $A/(c_1, c_2, c_3, c_4)$ and apply II.1.5.

Finally statement 4) follows since by 3) and II.2.4 we have A Noetherian. Then by II.2.5, A is a domain.

Recall that when the point scheme Γ of a quantum $\mathbb{P}^3$ is finite then Γ consists of 20 points counted with multiplicity. The following theorem gives the multiplicities of the 7 distinct points in Example III.2.12. We do not know the multiplicities of the points in Example III.2.11.

Theorem III.2.13. *The point scheme Γ consists of 7 distinct points, $\{p_1, \ldots p_7\}$ and the automorphism τ has 3 orbits of order 2 and fixes p_7. The points are given by: $p_1 = ([1,0,0,i],[1,0,0,i])$, $p_2 = ([1,0,0,-i],[1,0,0,-i])$. The points p_3, p_4 are given by $([1, a_2, a_3, 0], [1, a_2^{-1}, \dfrac{-a_3}{a_2}, 0])$, where a_2 satisfies $x^2 - bcx + 1 = 0$ and $a_3 =$*

$\frac{-a_2c}{a_2-1}$. *And finally,* $p_5 = ([1,-1,0,0],[0,0,0,1])$, $p_6 = ([0,0,0,1],[1,-1,0,0])$ *and* $p_7 = ([0,0,1,0],[0,0,1,0])$. *Let* $E = \pi_1(\Gamma)$ *where* $\pi_1 : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ *is the projection on the first factor. The multiplicities of the points are given by* $m(p_1) = m(p_2) = m(p_3) = m(p_4) = 2$, $m(p_5) = m(p_6) = 3$ *and* $m(p_7) = 6$.

Proof. Let $([a_1,a_2,a_3,a_4],[b_1,b_2,b_3,b_4])$ denote coordinates on $\mathbb{P}^3 \times \mathbb{P}^3$ so that the homogeneous coordinate ring of Γ is $\mathbb{C}[a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4]/J$ where J is the ideal generated by the $r_i(a,b)$ with r_i a relation of A. We first consider the basic affine open neighborhood U of p_7 given by setting $a_3 = 1, b_3 = 1$. In the coordinate ring of $U \cap \Gamma$ we have the relations:

$$\begin{aligned} a_1b_2 + a_2b_1 - bca_2b_2, &\qquad a_1b_4 - a_4b_1 + a_2b_4 - a_4b_2, \\ a_1 + b_1 + a_2 + b_2, &\qquad a_2 + b_2 - ca_2b_2, \\ a_1b_1 + a_4b_4 - a_2b_2, &\qquad b_4 - a_4, \end{aligned}$$

Hence $b_4 = a_4$, $b_1 = -a_1 - a_2 - b_2$. Making these substitutions and simplifying we have:

$$a_1a_2 + a_2^2 + (a_2 - a_1 + bca_2)b_2 = 0 \tag{1}$$

$$(a_1 + a_2)a_4 = 0 \tag{2}$$

$$a_2 + (1 - ca_2)b_2 = 0 \tag{3}$$

$$a_1^2 + a_1a_2 - a_4^2 + (a_1 + a_2)b_2 = 0 \tag{4}$$

Let $\mu = \frac{c}{2+bc}$ then $\mu(1) + (3) + \mu(4) = 0$ yields $b_2 = -\mu((a_1+a_2)^2 - a_4^2) - a_2$. If we substitute this expression for b_2 and change variables via $x = a_1 + a_2$, $y = a_2$,

$z = a_4$ then the coordinate ring of E in the affine neighborhood U is given by

$$\frac{k[x, y, z]}{(xz, cy^2 - \mu(1 - cy)(x^2 - z^2), x^2 - 2xy - z^2 - \mu x^3)}.$$

This algebra has exactly 3 maximal ideals corresponding to the points p_7 and p_3, p_4. Let M_3, M_4 denote the maximal ideals corresponding to the points p_3, p_4 respectively. Note that $x = a_1 + a_2$ is not in M_i, for $i = 3, 4$. We now localize at x. Then x becomes a unit so we have $z = 0$, giving the ring

$$\frac{k[x, y]}{(cy^2 - \mu(1 - cy)x^2, x(x - 2y - \mu x^2))}.$$

Again x is invertible so $x - 2y - \mu x^2 = 0$. So $y = \dfrac{x(1 - \mu x)}{2}$. Substituting this expression for y in $cy^2 - \mu(1 - cy)x^2$ implies that $x^2(c - 4\mu - \mu^2 cx^2)$ is now a relation. Since x is invertible, we have $c - 4\mu - \mu^2 cx^2 = 0$. This leaves us with the semi-local ring

$$\frac{k[x]}{(c - 4\mu - \mu^2 cx^2 = 0)}.$$

This is a 2-dimensional ring whose two maximal ideals correspond to the points p_3 and p_4. Hence $m(p_3) = m(p_4) = 1$.

Next we consider the affine open neighborhood V of the points p_1, p_2 given by setting $a_4 = b_4 = 1$. Notice that V intersects Γ at p_1, p_2. In the neighborhood V, the relations in the coordinate ring of $V \cap \Gamma$ are:

$$a_1b_2 + a_2b_1 - bca_2b_2, \tag{5}$$

$$a_1 - b_1 + a_2 - b_2, \tag{6}$$

$$a_1b_3 + a_3b_1 + ca_2b_2, \tag{7}$$

$$a_2b_3 + a_3b_2 - ca_2b_2, \tag{8}$$

$$a_1b_1 + 1 - a_2b_2, \tag{9}$$

$$a_3 - b_3. \tag{10}$$

So we have $b_3 = a_3, b_2 = a_1 + a_2 - b_1$, and $a_2b_2 = a_1b_1 + 1$. Notice that adding (7) and (8) results in the relation $a_1b_3 + a_3b_1 + a_2b_3 + a_3b_2$ so we may replace (7) by $a_1b_3 + a_3b_1 + a_2b_3 + a_3b_2$. With these substitutions we get:

$$a_1(a_1 + a_2) + ba_1a_3 + (-a_1 + a_2 + ba_3)b_1 = 0, \tag{11}$$

$$(a_1 + a_2)a_3 = 0 \tag{12}$$

$$(a_1 + 2a_2)a_3 + (-ca_1 - a_3)b_1 - c = 0 \tag{13}$$

$$a_2(-a_1 - a_2) + (a_1 + a_2)b_1 + 1 = 0 \tag{14}$$

Changing variables via $x = a_1, y = a_1 + a_2, z = a_3, w = b_1$ gives the following relations:

$$xy + bxz + (y - 2x + bz)w \tag{15}$$

$$yz \tag{16}$$

$$(2y - x)z + (-cx - z)w - c \tag{17}$$

$$-(y - x)y + yw + 1 \tag{18}$$

Let I be the ideal generated by these relations. Multiplying (18) by z and using (16) we see that $z \in I$. Hence we have $xy + (y-2x)w$, $xw+1$, and $-(y-x)y + yw + 1$ in I. So $xy + (y-2x)w + 2(xw+1) - (-(y-x)y + yw + 1) = y^2 + 1$ implies $y^2 + 1 \in I$. Also $y(-(y-x)y + yw + 1) \in I$ implies $w - 2y + x \in I$. Putting all of this together we get that the coordinate ring of $V \cap \Gamma$ is

$$\frac{k[x,y]}{(y^2+1, -2-4xy+2x^2, 2xy-x^2+1)}.$$

This is a 4-dimensional algebra with basis $\{1, x, y, xy\}$ which is a semilocal ring with 2 maximal ideals M_1, M_2 corresponding to p_1 and p_2. Localize at M_1. Then $y - i \notin M_1$ so $y^2 + 1 = (y+i)(y-i) = 0$ implies that $y + i = 0$ in the localization. The local ring is given by

$$k[x]/(x^2 + 2ix - 1),$$

a 2-dimensional ring. Hence the multiplicity of $p_1 = 2$. Similarly, localizing at M_2 implies the multiplicity of p_2 is 2.

Finally we need to compute $m(p_5)$ and $m(p_6)$. Since the automorphism τ of Γ permutes p_5 and p_6, we need only compute the multiplicity of p_5. Let W be the affine open neighborhood of p_5 given by $a_2 = b_4 = 1$. Notice that $W \cap \Gamma = \{p_5\}$. The relations in the coordinate ring of $\Gamma \cap W$ are given by

$$\begin{array}{cc} a_1b_2 + b_1 - bcb_2, & a_1 - a_4b_1 + 1 - a_4b_2 \\ a_1b_3 + a_3b_1 + cb_2, & b_3 + a_3b_2 - cb_2 \\ a_1b_1 + a_4 - b_2, & a_3 - a_4b_3 \end{array}$$

Solving for b_1 and b_3 we have:

$$b_1 = (bc - a_1)b_2 \qquad\qquad b_3 = (c - a_3)b_2$$

Making these substitutions we have:

$$a_1 + 1 - a_4(bc - a_1 - 1)b_2 = 0 \tag{19}$$

$$(-2a_1a_3 + ca_1 + bca_3 + c)b_2 = 0 \tag{20}$$

$$a_4 + (a_1(bc - a_1) - 1)b_2 = 0 \tag{21}$$

$$a_3 - a_4(c - a_3)b_2 = 0 \tag{22}$$

Now (19)-b(22) implies that $(a_1 - ba_3 + 1)(1 + a_4b_2) = 0$. We now localize at the maximal ideal $M_5 = (a_1+1, a_3, a_4, b_2)$. Noting that $1+a_4b_2 \notin M_5$, we get $a_1 = ba_3 - 1$. Making this substitution results in,

$$2b_2a_3(1 + bc - ba_3) = 0 \tag{23}$$

$$b_2(-2 - bc + 2ba_3 + b^2ca_3 - b^2a_3^2) + a_4 = 0 \tag{24}$$

$$a_3 - cb_2a_4 + b_2a_3a_4 = 0 \tag{25}$$

$$b(a_3 - cb_2a_4 + b_2a_3a_4) = 0 \tag{26}$$

Since $bc \neq -1$, the element $1 + bc - ba_3$ is not in M_5. Therefore $b_2a_3 = 0$. Using (24), we have $a_4 = (2 + bc)b_2$. Using this to eliminate a_4 we have:

$$-b(2cb_2 + bc^2b_2^2 - a_3) = 0 \qquad\qquad -2cb_2^2 - bc^2b_2^2 + a_3 = 0$$

Hence $a_3 = c(2+bc)b_2^2$, so that we have the relation $b_2^3(-1+bcb_2^2)$. Now $-1+bcb_2^2 \notin M_5$ so we have $b_2^3 = 0$. Hence the local ring is given by

$$\frac{k[b_2]}{(b_2^3)},$$

a 3-dimensional ring. So the multiplicity of the point p_5 is 3. Whence the multiplicity of p_6 is also 3.

Finally the multiplicity of p_7 is $20 - \sum_{i=1}^{6} m(p_i) = 6$. This completes the proof.

$\square$

CHAPTER IV

GEOMETRY OF QUANTUM $\mathbb{P}^3$'s.

IV.1. Fat point modules

In his paper [1] Artin defined the notions of fat point and fat point module.

Definition IV.1.1. Let A be an $\mathbb{N}$-graded k-algebra.

1) A fat point is an object $F \in \operatorname{Proj} A$ which is an equivalence class of graded GK-critical A-modules of GKdim 1 and multiplicity $m > 1$.

2) A fat point module F is a 1-critical graded A-module of multiplicity $m > 1$.

In 2), 1-critical means GKdim $= 1$ and every proper quotient is finite-dimensional and multiplicity m means that $\dim_k F_n = m$ for all n sufficiently large.

Let R be a graded k-algebra and suppose that R is finite over its center $Z(R)$. Then one expects there to be many fat point modules and indeed this is the case by the following theorem.

Theorem IV.1.2. *Suppose A is a graded k-algebra which is finite over a graded central subalgebra C. Suppose* $\dim \operatorname{Proj} C \geq 1$*. If A has a finite point scheme then A has infinitely many fat point modules.*

Proof. Let $p \in \operatorname{Proj} C$ be a closed point. We construct a graded A-module by defining $\hat{p} = C/p \otimes_C A$. Now A finite over C implies $\operatorname{GKdim}(\hat{p}) = \operatorname{GKdim}(C/p) = 1$ so that $\hat{p}$ has a composition series with 1-critical factors which are necessarily either fat point modules or point modules. As p varies through $\operatorname{Proj} C$ the A-modules $\hat{p}$, having different central actions, must be nonisomorphic as A-modules. By [23] (p. 9), the 1-critical factors in a composition series of a GK dimension 1 module are uniquely determined. Therefore since A has only finitely many point modules it follows that there are infinitely many fat point modules. $\square$

We wish to determine the fat point modules of the Shelton-Tingey algebra A. We first discuss how one sets up the computation in order to calculate the fat points. A fat point module of multiplicity $m > 1$ is equivalent in $\operatorname{Proj} A$ to a module F with constant Hilbert series, $H_F(t) = m$. Therefore a fat point module is determined by the data of a collection of linear maps $\varphi_i : A_1 \to \operatorname{Hom}(F_i, F_{i+1})$ for $i \geq 0$. By choosing fixed bases for the m-dimensional spaces F_i, we may identify $\operatorname{Hom}(F_i, F_{i+1})$ with the space $M_m(k)$ of $m \times m$ matrices over k. Since we are dealing with right modules, matrices always act on the right.

The collection of maps $(\varphi_i)_{i \geq 0}$ satisfies the further condition that whenever $r = \sum_{i,j} c_{ij} x_i \otimes x_j$, $c_{ij} \in k$, $x_i, x_j \in A_1$ is a relation of A then $\sum_{i,j} c_{ij} \varphi_k(x_i) \varphi_{k+1}(x_j) = 0$ in $\operatorname{Hom}(F_k, F_{k+2})$. Thus these conditions yield equations on the matrix entries of the φ_k. In general, a solution $(\varphi_i)_{i \geq 0}$ of the equations will not yield a fat point module because we further require that fat point modules give simple objects in the category

Proj A. Hence the matrices giving the φ_i cannot eventually all have the same block form. This requirement further restricts the possible cases that have to be considered.

In order to compute the fat point modules we fix the following notation. Suppose F is any $\mathbb{N}$-graded right A-module. Let A_i, B_i, C_i, D_i, $i = 1, 2, 3, 4$ denote the action of the linear element $x_i \in A_1$ on F_0, F_1, F_2, F_3, respectively. In other words, in the notation of the previous paragraph, $\varphi_0 : A_1 \to \mathrm{Hom}(F_0, F_1)$ is the linear map defined by sending $x_i \mapsto A_i$. Similarly φ_1 corresponds to the B_i, etc..

Finally, since twisting an algebra A does not change $\mathrm{Gr\,Mod}$, in order to find the fat point modules for A, it suffices to compute the fat point modules for A^σ for any $\sigma \in \mathrm{Aut}_{\mathrm{Gr}}(A)$.

Fat Point Modules for the twist A^{d_3}.

In this section we will determine some of the fat point modules for the twist of the Shelton-Tingey example, given by the automorphism d_3. Recall that this twist has relations given by

$$r_1 = x_3 \otimes x_1 + x_1 \otimes x_3 - ix_2 \otimes x_2 \qquad r_2 = x_4 \otimes x_1 + x_1 \otimes x_4$$

$$r_3 = x_4 \otimes x_2 + x_2 \otimes x_4 - ix_3 \otimes x_3 \qquad r_4 = x_3 \otimes x_2 - x_2 \otimes x_3$$

$$r_5 = x_1 \otimes x_1 + x_3 \otimes x_3 \qquad r_6 = x_2 \otimes x_2 + x_4 \otimes x_4$$

Recall this algebra has a central subring $C = k[x_1^2, x_2^2, \Omega_1, \Omega_2]$ where $\Omega_1 = (x_1x_2)^2 + (x_2x_1)^2$ and $\Omega_2 = (x_3x_4)^2 + (x_4x_3)^2$. Also recall that A is a free C-module. Given a fat point module F, the annihilator $\mathrm{Ann}_A(F)$ is a prime ideal in A by [4], Proposition

2.30 (vi). Consider the intersection $\zeta = C \cap \mathrm{Ann}_A(F)$. We will now prove that, in this setting, ζ is a closed point of $\mathrm{Proj}\, C$.

Lemma IV.1.3. *Suppose $C \subset A$ is an extension of graded rings and suppose C is commutative. Suppose P is a graded prime ideal of A. Let $Q = C \cap P$. Then Q is a prime ideal of C.*

Proof. We first note that Q is a proper ideal in C. If not then Q contains 1 which implies $P = A$. It is also trivial to see that Q is an ideal. We need to see that Q is prime.

Suppose $Q \supset IJ$ for some ideals I, J in C. Then $P \supset (AI)(JA)$. Since I, J are in C, AI and JA are two-sided ideals in A. Since P is prime, we may assume that $P \supset AI$. Now we have $I \subset AI \cap C \subset P \cap C = Q$. Hence Q is prime. □

Let F be a fat point module. We may assume F is cyclic and generated in degree 0, say $F = v_0 A$, $0 \neq v_0 \in F_0$.

Lemma IV.1.4. *Let $\zeta = C \cap \mathrm{Ann}_A(F)$. Then $\zeta = \mathrm{Ann}_C(v_0)$.*

Proof. Let $x \in \zeta$ then $v_0 x = 0$ so $x \in \mathrm{Ann}_C(v_0)$. Conversely, suppose $y \in \mathrm{Ann}_C(v_0)$ and let $w = v_0 a \in F$ for some $a \in A$. Then

$$\begin{aligned} wy &= (v_0 a)y \\ &= (v_0 y)a \\ &= 0. \end{aligned}$$

□

Proposition IV.1.5. *Suppose A is a finitely generated module over a connected graded central subalgebra C. Let F be a fat point module for A. Let $\zeta = C \cap \operatorname{Ann}_A(F)$. Then ζ is a closed point in* $\operatorname{Proj} C$.

Proof. Since $\operatorname{Ann}_A(F)$ is prime in A, by IV.1.3 we have that ζ is a graded prime ideal of C. By IV.1.4, C/ζ embeds in F. So since $\operatorname{GKdim} F = 1$ we have $\operatorname{GKdim} C/\zeta \leq 1$. If $\operatorname{GKdim} C/\zeta = 1$ then ζ is a graded maximal ideal, ie a closed point of $\operatorname{Proj} C$. Otherwise $\operatorname{GKdim} C/\zeta = 0$ in which case ζ is the irrelevant ideal $C_+ = C_{\geq 1}$.

Suppose $\zeta = C_+$. Then since A is a finitely generated C-module, we may choose homogeneous generators $\{a_1, \ldots a_n\}$ and write $A = \sum_{i=1}^{n} Ca_i$. Then we would have $F = v_0.A = v_0.(\sum_{i=1}^{n} Ca_i) = \sum_{i=1}^{n} k(v_0 a_i)$ which contradicts the fact that $\operatorname{GKdim} F = 1$. This completes the proof. □

We are going to consider the affine subscheme of $\operatorname{Proj} C$ given by all ζ which do not contain x_1^2.

We need the following general result about graded prime ideals in a weighted polynomial ring.

Proposition IV.1.6. *Let $R = k[x_1, x_2, x_3, x_4]$ be a commutative polynomial ring with* $\deg(x_1) = \deg(x_2) = 1$ *and* $\deg(x_3) = \deg(x_4) = 2$. *Then the maximal graded prime ideals of R are given by*

1) $(\alpha x_1 - x_2, \beta x_1^2 - x_3, \gamma x_1^2 - x_4)$

2) $(\alpha x_2 - x_1, \beta x_2^2 - x_3, \gamma x_2^2 - x_4)$

3) $(x_1, x_2, \alpha x_3 - \beta x_4)$

where $\alpha, \beta, \gamma \in k$.

Proof. Consider the second Veronese subalgebra $R^{(2)} = k[x_1^2, x_1x_2, x_2^2, x_3, x_4]$. Then $R \supset R^{(2)}$ is an integral extension. So the lying over theorem implies that for each graded prime ideal P in $R^{(2)}$ there exists a graded prime ideal Q of R lying over P. Furthermore, in this case, the graded prime Q is unique. This follows from Lemma 2.4 in [16].

Let P denote a maximal graded prime ideal of $R^{(2)}$, and let Q be the graded prime ideal of R lying over P.

Case 1: $x_1^2 \notin P$. Then $P = (\alpha x_1^2 - x_1x_2, \alpha' x_1^2 - x_2^2, \beta x_1^2 - x_3, \gamma x_1^2 - x_4)$, with $\alpha' = \alpha^2$. Now $\alpha x_1^2 - x_1x_2 = x_1(\alpha x_1 - x_2)$ and by our assumption, $x_1 \notin Q$. Therefore $\alpha x_1 - x_2 \in Q$. Let $Q = (\alpha x_1 - x_2, \beta x_1^2 - x_3, \gamma x_1^2 - x_4)$ then Q is a graded prime ideal of R and clearly lies over P so it is the unique prime over P.

Case 2: $x_2^2 \notin P$. This case is exactly the same as Case 1.

Case 3: $x_1 \in P$, $x_2 \in P$.

Note that $R^{(2)}/(x_1, x_2) \cong k[x_3, x_4]$ and maximal graded primes of $k[x_3, x_4]$ have the form $(\alpha x_3 - \beta x_4)$. So the unique graded prime lying over P is $Q = (x_1, x_2, \alpha x_3 - \beta x_4)$.

Cases (1)-(3) are exhaustive so this completes the proof.

□

Definition IV.1.7. Let M be a graded A-module and $C \subset A$ a graded central subalgebra. We call $\zeta = C \cap \text{Ann}_A(M)$ the central character associated to C and M.

When we compute fat point modules we will also compute their central characters. For the twisted algebra A^{d_3}, the following proposition describes the maximal graded primes for the central subalgebra $k[x_1^2, x_2^2, \Omega_1, \Omega_2]$. These will turn out to be the central characters of the fat point modules.

Proposition IV.1.8. *Let $k[x_1^2, x_2^2, \Omega_1, \Omega_2]$ be the polynomial subalgebra of the twist A^{d_3} where $\Omega_1 = (x_1x_2)^2 + (x_2x_1)^2$ and $\Omega_2 = (x_3x_4)^2 + (x_4x_3)^2$. Then the maximal graded prime ideals are given by:*

1) $(\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)$

2) $(\alpha x_2^2 - x_1^2, \beta x_2^4 - \Omega_2, \gamma x_1^4 - \Omega_2)$

3) $(x_1^2, x_2^2, \alpha\Omega_1 - \beta\Omega_2)$

where $\alpha, \beta, \gamma \in k$.

Proof. We first regrade so that the degrees of the generators of C are (1,1,2,2) respectively. Then we apply IV.1.6. □

Assumption 1: x_1^2 acts injectively on the fat point module F.

We use the notation A_i, B_i, C_i for $i = 1, 2, 3, 4$ for the action of x_i on F_0, F_1, F_2 respectively. Also we let I denote the identity matrix whose size is determined in context. Thus under assumption 1, we may choose bases so that A_1, B_1 and C_1 act as the identity matrix. Then the relations become:

$$r_1 = A_3 + B_3 - iA_2B_2 \qquad r_2 = A_4 + B_4$$

$$r_3 = A_4B_2 + A_2B_4 - iA_3B_3 \qquad r_4 = A_3B_2 - A_2B_3$$

$$r_5 = I + A_3A_3 \qquad r_6 = A_2B_2 + A_4B_4$$

Hence $B_4 = -A_4$, $B_3 = -A_3^{-1}$, and $B_2 = -B_3A_2B_3$. The analogous formulas for the C_i are given by $C_4 = -B_4 = A_4$, $C_3 = -B_3^{-1} = A_3$, and $C_2 = -C_3B_2C_3 = A_2$. If we can solve the relations for the $A's$ and $B's$ then since $C_i = A_i$ for $1 \leq i \leq 4$, the truncated module of length 3 extends uniquely to a fat point module. Therefore to find all fat point modules in this case, it suffices to solve the relations for the $A's$ and $B's$. We eliminate B_4, A_3 and B_2 which leaves us with:

$$-B_3^{-1} + B_3 + i(A_2B_3)^2 \tag{1}$$

$$-A_4B_3A_2B_3 - A_2A_4 + iI \tag{2}$$

$$-(A_2B_3)^2 - A_4^2 \tag{3}$$

Multiplying (1) by B_3 on the left and the right gives the two equations

$$B_3^2 + iB_3(A_2B_3)^2 - I = 0 \qquad B_3^2 + i(A_2B_3)^2B_3 - I = 0$$

So we have $B_3(A_2B_3)^2 = (A_2B_3)^2B_3$, and from (3) we get $B_3A_4^2 = A_4^2B_3$. Notice that B_3 is invertible, so we also get $(B_3A_2)^2 = (A_2B_3)^2$. Hence

$$A_2(A_4)^2 = -A_2(A_2B_3)^2 = -A_2(B_3A_2)^2 = A_4^2A_2.$$

Finally, since we must solve $B_4C_2 + B_2C_4 - iB_3C_3 = 0$, we also have the relation $-A_4A_2 - B_3A_2B_3A_4 + iI$. We summarize the equations we will use below:

$$r_1 = B_3^2 - iA_4^2B_3 - I \qquad r_2 = (A_2B_3)^2 + A_4^2$$

$$r_3 = B_3A_2B_3A_4 + A_4A_2 - iI \qquad r_4 = A_4B_3A_2B_3 + A_2A_4 - iI$$

Multiplicity 2 fat point modules

In this subsection we determine all of the multiplicity 2 fat point modules under the assumption that x_1 acts injectively. We consider cases based on the Jordan block form of A_4. Changing basis will not change the assumption that A_1, B_1 are the 2 x 2 identity matrix. If A_4 is diagonalizable with A_4^2 having distinct eigenvalues then the fact that A_2 and B_3 both commute with A_4^2 immediately implies A_2, B_3 and A_4 are all diagonal. This implies the corresponding module is not simple, it has a point module as a submodule. Similarly if $A_4 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ with $a \neq 0$, then we again have that A_2 and B_3 would be upper triangular. Hence any solution would have a point module as a submodule. Note also that from r_3, A_4 cannot be the zero matrix.

There are three remaining cases:

Case 1: $A_4 = aI$, $a \neq 0$,

Case 2: $A_4 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $a \neq 0$,

Case 3: $A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We begin with;

Case 1: $A_4 = aI$, $a \neq 0$.

Since A_4 is a scalar matrix, we may assume B_3 is in Jordan canonical form. We also note B_3 satisfies the polynomial $p(x) = x^2 - ia^2x - 1$ which follows from r_1. If $a^4 \neq 4$ then $p(x)$ has distinct roots so that B_3 is diagonalizable.

Case 1 (a): Assume $a^4 \neq 4$.

If B_3 is scalar then we could put A_2 in upper triangular form without changing A_4 or B_3, which would yield a nonsimple solution. Hence we may assume that $B_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}$ where a_3, d_3 are the distinct roots of $p(x)$. Thus $a_3d_3 = -1$ and $a_3 + d_3 = ia^2$. Let $P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda\mu \neq 0$. This is a nontrivial change of basis matrix which fixes the diagonal matrices B_3 and A_4. Write $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. If $c_2 = 0$ then we would have a nonsimple solution, so we may assume $c_2 \neq 0$. Now conjugating by P with $\lambda = 1$ and $\mu = c_2^{-1}$ we may assume $c_2 = 1$.

We have satisfied the relation r_1. Note that r_3 and r_4 are the same equation since A_4 is scalar. Examining the matrix entries in r_2 implies that $a_2 = \dfrac{i}{a(a_3^2+1)}$ and $d_2 = \dfrac{i}{a(d_3^2+1)}$. However b_2 is still undetermined, it remains to solve r_2. The matrix entries of r_2 are:

$$\begin{pmatrix} a_3(a_2^2a_3 + b_2d_3) + a^2 & b_2d_3(a_2a_3 + d_2d_3) \\ a_3(a_2a_3 + d_2d_3) & d_3(a_3b_2 + d_3d_2^2) + a^2 \end{pmatrix}$$

Direct computation shows that $a_2a_3 + d_2d_3 = 0$. Solving the (1,1) entry of the above matrix for b_2 gives $b_2 = a^2 + a_2^2a_3^2$. The (2,2) entry of the above gives $b_2 = a^2 + d_2^2d_3^2$. These are consistent because $(a_2a_3)^2 - (d_2d_3)^2 = 0$. This ends Case 1 (a).

Case 1 (b): $a^4 = 4$.

In this case the polynomial $p(x)$ has one root namely $\pm i$. So that B_3 is either $\pm iI$ or $\begin{pmatrix} \pm i & 1 \\ 0 & \pm i \end{pmatrix}$ where the signs are the same. The case where B_3 is scalar implies that we can change basis to get A_2 in upper triangular form so we must have $B_3 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, with $b = \pm i$. The stabilizer of B_3 in this case consists of all matrices of the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ with $x \neq 0$. Write A_2 as above, then we may assume that $c_2 \neq 0$, since if $c_2 = 0$, then we have a point module as a submodule. Let $P = \begin{pmatrix} 1 & -\frac{a_2}{c_2} \\ 0 & 1 \end{pmatrix}$. Then after conjugation by P we may assume $a_2 = 0$. The relation r_3 is linear in the entries of A_2, solving for the entries of A_2 we have $c_2 = \pm a^{-1}$ and $d_2 = ia^{-1}$. It remains to solve for b_2 which we can do by looking at r_2 which implies $b_2 = \pm a^3$. The choice of signs is determined by choosing the same sign as b in B_3, and these are all solutions. This completes Case 1 (b).

Case 2: A_4 has distinct eigenalues a and $-a$, with $a \neq 0$.

B_3 satisfies the polynomial $p(x)$ given above, so again we consider the two possibilities: (a) $p(x)$ has two distinct roots or (b) $p(x)$ has exactly one root.

Case 2 (a): $a^4 \neq 4$.

B_3 is diagonalizable with eigenvalues coming from the set $\{r_1, r_2\}$ of roots of $p(x)$. If B_3 is scalar then we may change basis so that $A_4 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. Then the (2,1) entry of r_3 is $ac_2(a_3^2 - 1)$. If $c_2 = 0$ then the solution cannot be simple, so we must have $a_3 = \pm 1$. But then ± 1 is a root of $p(x)$ which implies $a = 0$, a contradiction. Therefore we have $B_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}$ where a_3, d_3 are the distinct roots of $p(x)$. Consider the change of basis matrix $P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Conjugation by P fixes B_3 and allows us to consider two cases for A_4, namely

$$\text{(i)} \quad A_4 = \begin{pmatrix} a_4 & b_4 \\ 1 & d_4 \end{pmatrix} \qquad \text{(ii)} \quad A_4 = \begin{pmatrix} a_4 & b_4 \\ 0 & -a_4 \end{pmatrix}.$$

First consider case (i). Since A_4^2 is a scalar matrix we immediately have $d_4 = -a_4$. Recalling that $a_3 d_3 = -1$, the matrix entries of r_3 are given by:

$$\begin{pmatrix} i - a_2a_4 - a_2a_3^2a_4 - b_2 + b_4c_2 & 2a_4b_2 - a_2b_4 - a3^{-2}b_4d_2 \\ -a_2a_3^2 - 2a_4c_2 - d_2 & i + b_2 - b_4c_2 + a_4d_2 + a_3^{-2}a_4d_2 \end{pmatrix}$$

and the entries of r_4 are given by:

$$\begin{pmatrix} i - a_2a_4 - a_2a_3^2a_4 + b_2 - b_4c_2 & -2a_4b_2 - a_2a_3^2b_4 - b_4d_2 \\ -a_2 + 2a_4c_2 - a_3^{-2}d_2 & i - b_2 + b_4c_2 + a_4d_2 + a_3^{-2}a_4d_2 \end{pmatrix}$$

Clearing denominators and adding the off diagonal entries yields: $a_4b_2(1+a_3^2) = 0, a_4c_2(1+a_3^2)$. If $a_4 = 0$ or if $a_3^2 = -1$ then the sum of the (1,1) entries implies that $2i = 0$ which is absurd. Therefore we have $b_2 = c_2 = 0$. Then r_2 implies $d_2 = -a_2a_3^2$ and that $i - a_2a_4 - a_2a_3^2a_4 = 0$. From r_4 we have $b_4 = -a_2^2a_3^2 - a_4^2$ and finally r_1 implies $1 - a_3^2 - ia_2^2a_3^3 = 0$. We summarize the solution below as:

$$A_2 = \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_4 & b_4 \\ 1 & -a_4 \end{pmatrix}$$

where

$$d_3 = -a_3^{-1} \qquad d_2 = -a_2a_3^2$$

$$b_4 = -a_2^2a_3^2 - a_4^2 \qquad 1 - a_3^2 - ia_2^2a_3^3 = 0$$

$$i - a_2a_4 + a_2a_3^2a_4 = 0.$$

Note that given a value for a_3 we can determine all other variables. However, $a_3 \notin \{\pm1, \pm i\}$. For recall that a_3 is a root of $p(x)$ so $a_3 = \pm1$ implies $a = 0$ and $a_3 = \pm i$ implies $a^4 = 4$, and we are not in either of these cases.

We now consider case (ii) in which $A_4 = \begin{pmatrix} a_4 & b_4 \\ 0 & -a_4 \end{pmatrix}$. Then the (1,1) entry from r_3 is $-2a_4c_2$. Since $a_4 = 0$ implies $a = 0$ we must have $c_2 = 0$ but then all of A_2, B_3, A_4 are upper triangular giving a nonsimple solution.

Case 2 (b): $a^4 = 4$.

In this case the polynomial $p(x)$ has exactly one root namely $\pm i$. Then we may put B_3 into Jordan canonical form and assume $B_3 = \begin{pmatrix} a_3 & 1 \\ 0 & a_3 \end{pmatrix}$ with $a_3 = \pm i$. The conjugation stabilizer of B_3 contains matrices of the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ with $x \neq 0$. Conjugation by a matrix of this type allows us to put A_4 into the following two possibilities:

$$\text{(i)} \quad A_4 = \begin{pmatrix} 0 & b_4 \\ c_4 & d_4 \end{pmatrix} \qquad \text{(ii)} \quad A_4 = \begin{pmatrix} a_4 & b_4 \\ 0 & d_4 \end{pmatrix}.$$

Consider possibility (ii). In this case the (1,1) entry of r_3 is $-2a_4c_2$. If $a_4 = 0$ then $a = 0$, a contradiction and if $c_2 = 0$ then this is a nonsimple solution. Hence we need only consider possibility (i). There are solutions in this case and they are summarized in:

$$A_2 = \begin{pmatrix} a_2 & \pm i a_2 \\ 0 & a_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} \pm i & 1 \\ 0 & \pm i \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & b_4 \\ c_4 & 0 \end{pmatrix},$$

where $b_4c_4 = \pm 2$, $a_2^2 = \pm 2$, and $a_2c_4 = \pm 1$. The signs are all determined consistently by the choice of sign in B_3. This gives precisely 4 solutions and concludes Case 2.

Case 3: $A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In this case B_3 satisfies the polynomial $q(x) = x^2 - 1$. Hence B_3 is diagonalizable and there are two possibilities for the Jordan form of B_3, namely (i) $B_3 = \pm I$ or (ii)

$B_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We first consider possibility (i). Since B_3 is scalar we may change basis and assume $A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Write $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then r_3 and r_4 give the same information and imply $c_2 = i$ and $d_2 = -a_2$. The stabilizer of A_4 contains matrices of the form $P = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$. Taking $\lambda = i$ and $\mu = -a_2$ and conjugating by P we may assume $a_2 = 0$ and so $d_2 = 0$. Then r_2 implies that $b_2 = 0$. We have the following solutions in this case:

$$A_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad B_3 = \pm I, \quad A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This finishes (i).

It remains to look at (ii). We write $A_4 = \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix}$. In this case $A_4^2 = 0$ and we consider two possibilities: (a) $c_4 = 0$ or (b) $c_4 \neq 0$. Conjugating by a diagonal matrix doesn't change B_3 and we can scale the off diagonal entries in A_4. Consider possibility (a). We may assume $b_4 = 1$ and then the sum of the off diagonal entries in r_3 implies $2i = 0$ which is absurd.

In (b) we have $d_4 = -a_4$ and we may change basis to get $c_4 = 1$. Then the difference of the (2,1) entries in r_3 and r_4 implies that $a_4 c_2 = 0$. If $a_4 = 0$ then the trace of r_3 is $2i$, a contradiction. If $c_2 = 0$ then the diagonal entries in r_2 are $a_2^2 + a_4^2 + b_4$ and $d_2^2 + a_4^2 + b_4$. Since $A_4^2 = 0$ we have $a_4^2 + b_4 = 0$ and then $a_2 = d_2 = 0$.

But then the sum of the off diagonal entries in r_3 is $2i$, a contradiction. This finishes Case 3 and concludes the analysis of the fat point modules of multiplicity 2 under Assumption 1.

We now wish to compute the central characters of the multiplicity 2 fat points. Under our assumption, x_1^2 acts injectively so that we are in the affine subscheme of $\operatorname{Proj} C$ consisting of the points $(\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)$, with $\alpha, \beta, \gamma \in k$. We will abbreviate these points as (α, β, γ). Recall that the automorphism induced on the fat points under assumption 1 has order 2. Thus given a fat point module F we need only compute the scalar matrices: $A_2B_2, B_2^2 + A_2^2, (A_3B_4)^2 + (A_4B_3)^2$ whose scalars yield α, β, γ respectively. We summarize the above analysis in the following.

Theorem IV.1.9. *Let A denote the twist of the Shelton-Tingey example by the automorphism d_3. Let $C = k[x_1^2, x_2^2, \Omega_1, \Omega_2]$ denote the central subalgebra given above. Let X denote the affine subscheme of* $\operatorname{Proj} C$ *given by*

$$\{(\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2) \mid \alpha, \beta, \gamma \in k\}.$$

Then the fat point modules for which x_1^2 acts injectively along with their central characters are:

1. *$\mathcal{F}_1$:*

$$A_2 = \begin{pmatrix} a_2 & b_2 \\ 1 & d_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}, \quad A_4 = aI,$$

where $a_2 = \dfrac{i}{a(1+a_3^2)}$, $d_2 = \dfrac{i}{a(1+d_3^2)}$, $b_2 = a^2 + a_2^2 a_3^2$ *give the entries of* A_2.

The entries of B_3 are the distinct roots of $x^2 - ia^2x - 1$, and $a^4 \notin \{0, 4\}$. The central character is given by $(a^2, \frac{2a^4-1}{a^2}, a^2(2-a^4)) = (\alpha, \beta, \gamma)$.

2. $\mathcal{F}_2$:

$$A_2 = \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \quad A_4 = aI,$$

where $b_2 = \pm a^3$, $c_2 = \pm a^{-1}$ and $d_2 = ia^{-1}$ give the entries of A_2. The diagonal entry in B_3, b is $\pm i$, and $a^4 = 4$. The choices of the signs are all determined by choosing the same sign as b. The central characters are $(\pm 2, \pm\frac{7}{2}, \mp 4) = (\alpha, \beta, \gamma)$ where the choice of sign is again determined by the sign of b.

3. $\mathcal{F}_3$:

$$A_2 = \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_4 & b_4 \\ 1 & -a_4 \end{pmatrix}.$$

The entries are determined by the following equations:

$$d_3 = -a_3^{-1} \qquad\qquad d_2 = -a_2a_3^2$$

$$b_4 = -a_2^2a_3^2 - a_4^2 \qquad\qquad 1 - a_3^2 - ia_2^2a_3^3 = 0$$

$$i - a_2a_4 - a_2a_3^2a_4 = 0$$

Set $a^2 = a_4^2 + b_4$ and assume $a^4 \neq 4$. The central character is $(a^2, -a^2(2 - a^4), \frac{1-2a^4}{a^2}) = (\alpha, \beta, \gamma)$.

4. $\mathcal{F}_4$:

$$A_2 = \begin{pmatrix} a_2 & \pm ia_2 \\ 0 & a_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & b_4 \\ c_4 & 0 \end{pmatrix},$$

where $b_4c_4 = \pm 2$, $a_2^2 = \pm 2$, $a_2c_4 = \pm 1$, *and* $b = \pm i$. *The signs are all determined by choosing the same sign as* b. *The central characters are* $(\pm 2, \pm 4, \mp\frac{7}{2}) = (\alpha, \beta, \gamma)$. *The signs are given by choosing the same sign as* b.

5. $\mathcal{F}_5$:

$$A_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad B_3 = \pm I, \quad A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The two central characters are $(0,0,0) = (\alpha, \beta, \gamma)$.

Proof. The formulas for the action of A on F_i were determined in the subsection Multiplicity 2 fat point modules. Recall that under the assumption that x_1^2 acts injectively, we have the formulas: $B_4 = -A_4$, $A_3 = -B_3^{-1}$, and $B_2 = -B_3A_2B_3$. It is now straightforward to compute the central characters for the fat point modules in the families F_1, F_2, F_4, F_5 in terms of the parameter a which gives the eigenvalue of the matrix A_4. The only exception is the family F_3. However notice that in this case the parameter a^2 is the unique eigenvalue of A_4^2. We compute as follows.

By multiplying out the matrices we compute that the central character is $(a_2d_2, a_2^2 + d_2^2, \frac{a_4^2}{a_3^2} + a_3^2a_4^2 - 2b_4)$. In this case we have used a_3 as a parameter so we want to write this character in terms of a_3 and then use a formula relating a_3 and a^2. We have $a_2d_2 = -a_2^2a_3^2 = a_4^2 + b_4 = a^2$. On the other hand, $-a_2^2a_3^2 = \frac{-a_3^2(1-a_3^2)}{ia_3^3} = \frac{a_3^2-1}{ia_3}$, so $a^2 = \frac{a_3^2-1}{ia_3}$.

Now we have $2 - a^4 = 2 - (\frac{a_3^2-1}{ia_3})^2 = \frac{a_3^4+1}{a_3^2}$ so $a_2^2 + d_2^2 = a_2^2(1 + a_3^4) =$

$\dfrac{(1-a_3^2)(1+a_3^4)}{ia_3^3} = -a^2(2-a^4)$. Finally we can write $\dfrac{2a^4-1}{a^2}$ in terms of a_3 as

$$\frac{2a^4-1}{a^2} = \frac{-i(2a_3^4-3a_3^2+2)}{a_3(a_3^2-1)}.$$

Calculating $\dfrac{a_4^2}{a_3^2} + a_3^2a_4^2 - 2b_4$ in terms of a_3, we find that

$$\frac{a_4^2}{a_3^2} + a_3^2a_4^2 - 2b_4 = \frac{2a^4-1}{a^2}.$$

So the character as stated for F_3 is correct. This finishes the proof.

We remark that in the 3-fold $\operatorname{Proj} C$, we have only found two affine curves worth of multiplicity 2 fat point modules.

□

Multiplicity 4 fat point modules

In this section we will determine most of the fat point modules of multiplicity 4 for the algebra A^{d_3} again working under the assumption that x_1^2 acts injectively. In $\operatorname{Proj} C$ this means we are considering the affine subscheme $X \subset \operatorname{Proj} C$ whose closed points have the form $\zeta = (\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)$. We first make the following definition.

Definition IV.1.10. Given a closed point $\zeta = (\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)$ in X define the k-algebra $A_\zeta = ((A/\zeta A)_{\{x_1^2\}})_0$. The notation $A_{\{x_1^2\}}$ is the localization of A at the central element x_1^2. We then take the subalgebra of degree 0 elements in this localization.

A_ζ is a finite-dimensional k-algebra and we compute its dimension as follows. We first note that $A/(x_1^2, \alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)A = A/(x_1^2, x_2^2, \Omega_1, \Omega_2)A$. Since $x_1^2, x_2^2, \Omega_1, \Omega_2$ is a regular sequence it follows from II.1.5 that $\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2$ is a regular sequence and x_1^2 is regular modulo ζ. Then the Hilbert series of $A/\zeta A$ is $\dfrac{(1-t^2)(1-t^4)^2}{(1-t)^4} = \dfrac{(1+t)^3(1+t^2)^2}{1-t}$ so that $\dim_k (A/\zeta A)_n = 32$ for $n \geq 8$. Since x_1^2 is regular modulo ζ, after localizing, (x_1^{2n}) defines an invertible linear map from $((A/\zeta A)_{\{x_1^2\}})_l$ to $((A/\zeta A)_{\{x_1^2\}})_{l+2n}$ for $l \in \mathbb{N}$. Hence $\dim((A/\zeta A)_{\{x_1^2\}})_l = 32$ for all $l \in \mathbb{N}$, in particular $\dim_k A_\zeta = 32$. This puts an upper bound on the multiplicity of fat points.

Proposition IV.1.11. *Let F be a fat point module for the algebra A^{d_3} and let m be the multiplicity of F. Then $m \leq 5$.*

Proof. Let ζ be the central character of F. Then $\zeta \in \operatorname{Proj} C$ by IV.1.1 so ζ is one of the graded prime ideals in IV.1.8. Then A_ζ is either a 16 or 32 dimensional ring. The fat point module F defines a simple module for A_ζ of dimension m. Then the Artin-Wedderburn theorem implies that the maximal dimension of a simple module for A_ζ is 5. □

We will now prove that generically the rings A_ζ for $\zeta \in X$ are semisimple by exhibiting a 3-parameter family of multiplicity 4 fat point modules. For a generic $\zeta \in X$, we find two non-isomorphic simple 4 dimensional representations of A_ζ. Since $\dim_k A_\zeta = 32$, the Artin-Wedderburn theorem implies that $A_\zeta \cong M_4(k) \times M_4(k)$ where $M_4(k)$ denotes the algebra of 4 x 4 matrices over k. Hence A_ζ is semisimple.

Theorem IV.1.12. *Define the multiplicity 4 A-module F by the formulas* $A_1 = I_4$ *and*

$$A_2 = \begin{pmatrix} L & 0 & 1 & 0 \\ 0 & -L & 0 & 1 \\ G & 0 & f^2L & 0 \\ 0 & G & 0 & -f^2L \end{pmatrix}, \quad A_3 = \begin{pmatrix} -f^{-1} & 0 & 0 & 0 \\ 0 & -f^{-1} & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix},$$

$$A_4 = \begin{pmatrix} \frac{i}{(1+f^2)L} & a_2 & 0 & 1 \\ b_1 & \frac{-i}{(1+f^2)L} & \frac{-b_1}{a_2} & 0 \\ 0 & G - a_2(1-f^2)L & \frac{i}{(1+f^2)L} & a_2 \\ \frac{-b_1}{a_2}(G - a_2L(1-f^2)) & 0 & b_1 & \frac{-i}{(1+f^2)L} \end{pmatrix},$$

where $L, f, G, a_2, b_1 \in k$ *with* $f \notin \{0, \pm i\}$*. Let* $B_1 = I_4$*,* $B_2 = -A_3^{-1}A_2A_3^{-1}$*,* $B_3 = -A_3^{-1}$*, and* $B_4 = -A_4$*. Let* $G = \frac{i}{f} - if + f^2L^2$*,* $H = (1+f^2)^2L^2$*, and*

$$p(x) = b_1fHx^2 - (f + L^2(i - b_1fL)(1 + f^2 - f^4 - f^6))x - b_1H(i - if^2 + f^3L^2).$$

Let a_2 *be a root of* $p(x)$*. Then for generic values of the parameters* b_1, f, L *the two values for* a_2 *give non-isomorphic fat point modules of multiplicity 4.*

Proof. Substituting the above matrices, without specializing a_2, into the relations of A, we are left with only one equation which says a_2 is a root of the quadratic $p(x)$. Therefore F is an A-module. For generic values of b_1, f, L, $p(x)$ will have distinct roots. We want to show that the two modules determined by the choice of roots for $p(x)$ are not isomorphic as A-modules. We denote these two modules by F and F'

and let A_i , A_i' denote the respective actions of x_i on the degree zero component of the module.

An isomorphism $F \cong F'$ defines an isomorphism $F_0 \cong F_0'$ which is just a change of basis in the degree 0 component. Hence an isomorphism implies the existence of an invertible matrix $X \in M_4(k)$ such that $XA_iX^{-1} = A_i'$ for $1 \leq i \leq 4$. Notice that $A_i = A_i'$ for $1 \leq i \leq 3$. The centralizer of A_3 is given by

$$C(A_3) = \{\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \mid P, Q \in \mathrm{GL}_2(k)\}$$

since $f^2 + 1 \neq 0$. Let $X = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in C(A_3)$. If $XA_2X^{-1} = A_2$ then the upper right 2 x 2 block of this equation implies immediately that $P = Q$. Furthermore $P \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} P^{-1} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$. Then since $L \neq 0$ this implies that P is a diagonal matrix. Now suppose that $XA_4X^{-1} = A_4'$ then looking at the upper 2 x 2 block of this equation it follows that $P \begin{pmatrix} 0 & 1 \\ \frac{-b_1}{a_2} & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{-b_1}{a_2} & 0 \end{pmatrix}$. But P is diagonal so the fact that the 1 in the (1,2) entry of this equation is fixed implies that P is scalar. Hence X is scalar. This is a contradiction because then $XA_4X^{-1} = A_4$. Therefore F and F' are not isomorphic as A-modules.

The only thing left to prove is that generically F is a simple object of $\operatorname{Proj} A$. If F isn't simple then it's an extension of either a point module or a multiplicity 2 fat point module. There are only 20 point modules and by IV.1.9, there are only two curves of multiplicity 2 fat points. Since generically the character of F is not the character of a point module or a multiplicity 2 fat point module, it follows that generically F is simple. □

The central characters of the modules F are given by:

Proposition IV.1.13. *Let F be a module given in IV.1.12. Let H be as in IV.1.12. Then for fixed values of the parameters b_1, f, L the central character of F and F' is:*

$$(\alpha x_1^2 - x_2^2, \beta x_1^4 - \Omega_1, \gamma x_1^4 - \Omega_2)$$

where $\alpha = \dfrac{i(1-f^2)}{f}$, $\beta = 2\alpha + H$, *and*

$$\gamma = \frac{1}{a_2 f^2 H}(a_2^2 b_1 H(1+f^4) + 2b_1 f H(i - if^2 + f^3L^2) +$$
$$a_2(-1 - 2b_1 f^2 L^3(1 - f^4 - f^6) - f^4(1 + 2b_1 L^3))).$$

Proof. Using the formulas in IV.1.12 we compute the matrices A_2B_2, $A_2^2+B_2^2$, $(A_3B_4)^2 + (A_4B_3)^2$. These are all scalar matrices and we get $\alpha I, \beta I, \gamma I$ respectively. The formulas for α, β, and γ follow. □

Notice that we can obtain a generic character $\zeta \in C$ for some values of the parameters b_1, f, L.

Conversely, suppose we fix a central character $\zeta = (\alpha, \beta, \gamma)$. Using the formulas in IV.1.13 for α and β, we fix two values of f and L. Using the formula for γ, we solve for b_1 in terms of a_2 and γ. Notice that the formula for γ is linear in b_1 so generically there is one solution for b_1. Substituting the values for f, L, and b_1 into $p(x)$, we get a quadratic polynomial whose coefficients are functions of α, β, and γ. Generically this polynomial will have two roots. Now the proof of IV.1.12 shows that the two modules determined by these two roots, along with the fixed values of b_1, f, and L, are not isomorphic. Therefore, generically, the ring A_ζ has two non-isomorphic simple 4-dimensional modules. By the Wedderburn theorem this implies A is semisimple and $A_\zeta \cong M_4(k) \times M_4(k)$.

IV.2. Line Modules

Throughout this section we will denote by A the Shelton-Tingey example and use R for a quantum $\mathbb{P}^n$.

Another notion central to the study of the noncommutative geometry of a quantum projective space is the concept of a line module.

Definition IV.2.1. Let R be a quantum $\mathbb{P}^n$ and L a graded right R-module. Then L is a line module if:

1) L is cyclic and generated in degree 1

2) $H_L(t) = \dfrac{1}{(1-t)^2}$

In the case of a quantum $\mathbb{P}^2$ the line modules are in one-to-one correspondence with the lines in the projective space $\mathbb{P}(R_1) \cong \mathbb{P}^2$. However in the case of a quantum $\mathbb{P}^3$ not every line in $\mathbb{P}(R_1)$ yields a line module. Given $l \subset \mathbb{P}(R_1)$, the associated module

R/lR may have the wrong Hilbert series. On the positive side, the line modules are parametrized by the following.

Theorem IV.2.2. *[19] Let $R = T(V^*)/I$ be a quantum $\mathbb{P}^3$. Let $\mathbb{G}^2(V^*)$ denote the Grassmannian scheme of codimension 2 subspaces of V^*. Then the line scheme is parametrized by*

$$\mathcal{L}^\perp = \{Q \in \mathbb{G}^2(V^*) \mid \dim_k(Q \otimes V^* + I_2) \leq 13\}.$$

One can consider incidence relations between point modules and line modules. If P is a point module and L is a line module, we say P lies on L and write $P \in L$ if there is a surjective homomorphism $L \twoheadrightarrow P$. There is another incarnation of the point scheme as

$$\mathcal{P} = \{p \in \mathbb{P}(V) \mid \dim_k(p^\perp \otimes V^* + I_2) = 15\}$$

and the line scheme has the isomorphic representation

$$\mathcal{L} = \{q \in \mathbb{G}_2(V) \mid \dim_k(q \otimes V \cap T_2^\perp \geq 3\}.$$

For $p \in \mathcal{P}$, let $P = N(p)$ denote the point module defined by p and for $q \in \mathcal{L}$, let $L = M(q)$ denote the line module defined by q. Then $P \in L$ if and only if $p \in \mathbb{P}(q)$. Given a point module P, let $\mathcal{L}_P$ be the subscheme of $\mathcal{L}$ consisting of line modules which cover P. We have:

Proposition IV.2.3. *[19] Let R be a quantum $\mathbb{P}^3$ and P a point module. Then $\mathcal{L}_P$ is nonempty, that is, every point module is covered by a line module.*

Generically $\dim(\mathcal{L}_p) = 0$ and in this case, $\mathcal{L}_p$ consists of 6 lines counted with multiplicity.

Given a point $p \in \mathbb{P}^3 = \mathbb{P}(V)$, the scheme of lines l passing through p is a subscheme of $\mathbb{G}_2(V)$. Let $\widetilde{\mathcal{L}}_p = \{l \in \mathbb{G}_2(V) \mid p \subset l\}$ be the subscheme of $\mathbb{G}_2(V)$ of lines passing through $p \in \mathbb{P}(V)$. Recall that $\mathbb{G}_2(V)$, for a 4-dimensional vector space V, can be embedded into $\mathbb{P}^5$ via the Plücker embedding. Let M_{ij} denote the ij minor of a generic 2×4 matrix and let $P_M = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$ be the Plücker relation. Let $S = \dfrac{\mathbb{C}[M_{ij}]}{\langle P_M \rangle}$ then $\mathbb{G}_2(V) = \operatorname{Proj} S$.

Lemma IV.2.4. *Let $p \in \mathbb{P}(V)$ where $\dim V = 4$. Then $\widetilde{\mathcal{L}}_p \cong \mathbb{P}^2$.*

Proof. By changing basis in V we may assume that $p = [1, 0, 0, 0]$. Suppose $p \subset l$ for some $l \in \mathbb{G}_2(V)$ then l is the row space of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & c & d \end{pmatrix}$$

so $M_{23} = M_{24} = M_{34} = 0$. Hence $\mathcal{L}_p \subset \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34})$.

Conversely, suppose $l \in \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34})$ and l is given as the row space of

$$M = \begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix}.$$

Then the matrix $\begin{pmatrix} b & c & d \\ b' & c' & d' \end{pmatrix}$ has rank 1 so M is row equivalent to

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a'' & b'' & c'' & d'' \end{pmatrix}.$$

Hence $l \in \mathcal{L}_p$. So we have $\mathcal{L}_p = \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34}) \cong \mathbb{P}^2$. □

In the Shelton-Tingey example A we have computed $\mathcal{L}_p$ for each of the 20 point modules. The following result describes $\mathcal{L}_p$ for $p \in \mathcal{P}$ as a subscheme of $\mathbb{G}_2(V)$. Let $F_1, \dots, F_n$ be homogeneous polynomials in the Plücker coordinates M_{ij}. We write $\mathcal{V}_{\mathbb{G}}(F_1, \dots, F_n)$ for Proj $\dfrac{S}{\langle F_1, \dots, F_n \rangle}$.

Theorem IV.2.5. *Let $A = T(V^*)/I$ be the Shelton-Tingey example. Then:*

1) $\mathcal{L}_{e_1} = \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34}, M_{12}^3 + iM_{13}^2 M_{14} - M_{12}M_{14}^2)$.

2) $\mathcal{L}_{e_2} = \mathcal{V}_{\mathbb{G}}(M_{13}, M_{14}, M_{34}, M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2)$.

3) $\mathcal{L}_{e_3} = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{14}, M_{24}, M_{34}^3 - M_{23}^2 M_{34} - iM_{13}^2 M_{23})$.

4) $\mathcal{L}_{e_4} = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{13}, M_{23}, M_{14}M_{24}^2 + iM_{14}^2 M_{34} - iM_{34}^3)$.

Let $p = [1, a_2, a_3, a_4]$ be one of the remaining 16 points. Then $\mathcal{L}_p$ is a nonreduced scheme of dimension 0 consisting of 3 line modules given by

- $L_1 = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{13}, M_{23}, M_{24} - a_2 M_{14}, M_{34} - a_3 M_{14})$
- $L_2 = \mathcal{V}_{\mathbb{G}}(M_{12} - \dfrac{a_2}{a_3} M_{13}, M_{14}, M_{23}, M_{24} + \dfrac{a_2 a_4}{a_3} M_{13}, M_{34} + a_4 M_{13})$

- $L_3 = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{14}, M_{23} - a_2M_{13}, M_{24}, M_{34} + a_4M_{13})$

where the multiplicities are given by $\text{mult}(L_1) = \text{mult}(L_3) = 1$ *and* $\text{mult}(L_2) = 4$.

Proof. By IV.2.4 and its proof, we immediately see that in the case of $\mathcal{L}_{e_i}$, the first three linear polynomials define the scheme $\widetilde{\mathcal{L}}_{e_i}$. For the point $p = [1, a_2, a_3, a_4]$, consider a line $l \in \mathbb{G}_2(V)$ passing through p. Then l is the row space of

$$M = \begin{pmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & x & y & z \end{pmatrix}.$$

Then we have $x = M_{12}$, $y = M_{13}$ and $z = M_{14}$. So it follows that

$$M_{23} = a_2M_{13} - a_3M_{12}, \quad M_{24} = a_2M_{14} - a_4M_{12}, \quad M_{34} = a_3M_{14} - a_4M_{13}.$$

So $\widetilde{\mathcal{L}}_p \subset \mathcal{V}_{\mathbb{G}}(M_{23} - a_2M_{13} + a_3M_{12}, M_{24} - a_2M_{14} + a_4M_{12}, M_{34} - a_3M_{14} + a_4M_{13})$. By IV.2.4 we know $\widetilde{\mathcal{L}}_p \cong \mathbb{P}^2$ and clearly $\mathcal{V}_{\mathbb{G}}(M_{23} - a_2M_{13} + a_3M_{12}, M_{24} - a_2M_{14} + a_4M_{12}, M_{34} - a_3M_{14} + a_4M_{13}) \cong \mathbb{P}^2$. Hence $\widetilde{\mathcal{L}}_p = \mathcal{V}_{\mathbb{G}}(M_{23} - a_2M_{13} + a_3M_{12}, M_{24} - a_2M_{14} + a_4M_{12}, M_{34} - a_3M_{14} + a_4M_{13})$. Since $\mathcal{L}_p = \mathcal{L} \cap \widetilde{\mathcal{L}}_p$ we next write down the equations describing $\mathcal{L}$ and then intersect with the scheme $\widetilde{\mathcal{L}}_p$.

Using MATHEMATICA we have the following 45 quartic polynomials in the Plücker coordinates M_{ij} which define the scheme $\mathcal{L}$.

$$-{M_{14}}^2\, M_{23}\, M_{24} - M_{13}\, M_{14}\, {M_{24}}^2 + i\, M_{13}\, M_{14}\, M_{23}\, M_{34} + M_{23}\, M_{24}\, {M_{34}}^2,$$

$$M_{13}\, M_{14}\, M_{23}\, M_{24} + i\, M_{12}\, M_{14}\, {M_{24}}^2 + i\, M_{23}\, {M_{24}}^2\, M_{34},$$

$$M_{12}\, M_{13}\, {M_{24}}^2 + i\, M_{14}\, {M_{24}}^3 - {M_{14}}^2\, M_{24}\, M_{34} + i\, M_{13}\, M_{14}\, {M_{34}}^2 + M_{24}\, {M_{34}}^3,$$

$$-i\, M_{13}\, {M_{14}}^2\, M_{23} + M_{12}\, {M_{14}}^2\, M_{24} + M_{14}\, M_{23}\, M_{24}\, M_{34},$$

$$i\, M_{12}\, {M_{14}}^2\, M_{23} + {M_{14}}^2\, {M_{24}}^2 + i\, {M_{14}}^3\, M_{34} - i\, {M_{14}}^2\, M_{23}\, M_{34}$$
$$-i\, M_{12}\, M_{23}\, {M_{34}}^2 - i\, M_{14}\, {M_{34}}^3,$$
$${M_{12}}^2\, M_{14}\, M_{24} + {M_{14}}^2\, M_{23}\, M_{24} + i\, M_{13}\, {M_{14}}^2\, M_{34} + M_{12}\, M_{23}\, M_{24}\, M_{34},$$
$$M_{13}\, M_{14}\, M_{23}\, M_{34} + i\, M_{12}\, M_{14}\, M_{24}\, M_{34} + i\, M_{23}\, M_{24}\, {M_{34}}^2,$$
$${M_{14}}^2\, {M_{23}}^2 + {M_{13}}^2\, {M_{24}}^2 - i\, {M_{13}}^2\, M_{23}\, M_{34} + i\, M_{14}\, {M_{24}}^2\, M_{34} - {M_{14}}^2\, {M_{34}}^2$$
$$+M_{14}\, M_{23}\, {M_{34}}^2 - {M_{23}}^2\, {M_{34}}^2 + {M_{34}}^4,$$

$${M_{13}}^2\, M_{23}\, M_{24} - i\, M_{12}\, M_{14}\, M_{23}\, M_{24} + i\, M_{14}\, M_{23}\, M_{24}\, M_{34}$$
$$-i\, {M_{23}}^2\, M_{24}\, M_{34} - M_{13}\, M_{14}\, {M_{34}}^2,$$
$$i\, {M_{13}}^2\, M_{14}\, M_{23} - M_{12}\, M_{13}\, M_{14}\, M_{24} - M_{13}\, M_{23}\, M_{24}\, M_{34},$$
$$-2\, M_{13}\, M_{14}\, M_{23}\, M_{24},$$
$$i\, M_{12}\, M_{13}\, M_{23}\, M_{34} - M_{14}\, M_{23}\, M_{24}\, M_{34} + i\, M_{13}\, M_{14}\, {M_{34}}^2,$$

$$-M_{12}\, M_{13}\, M_{23}\, M_{24} - i\, M_{14}\, M_{23}\, {M_{24}}^2 + M_{13}\, M_{14}\, M_{24}\, M_{34},$$
$$-i\, M_{12}\, M_{13}\, M_{14}\, M_{23} - {M_{14}}^2\, M_{23}\, M_{24} - i\, M_{13}\, {M_{14}}^2\, M_{34},$$
$$-M_{13}\, M_{14}\, {M_{23}}^2 - {M_{13}}^2\, M_{23}\, M_{24} - i\, M_{14}\, M_{23}\, M_{24}\, M_{34} + M_{13}\, M_{14}\, {M_{34}}^2,$$
$$-M_{13}\, M_{14}\, {M_{23}}^2 + i\, M_{12}\, M_{14}\, M_{23}\, M_{24} + i\, {M_{23}}^2\, M_{24}\, M_{34},$$

$$M_{12}\, M_{13}\, M_{23}\, M_{24} + i\, M_{14}\, M_{23}\, {M_{24}}^2 + M_{13}\, M_{14}\, M_{24}\, M_{34},$$
$$M_{13}\, M_{14}\, M_{23}\, M_{34} - i\, M_{12}\, M_{14}\, M_{24}\, M_{34} - i\, M_{23}\, M_{24}\, {M_{34}}^2,$$
$$-{M_{12}}^2\, M_{14}\, M_{23} - {M_{14}}^2\, {M_{23}}^2 - M_{12}\, {M_{14}}^2\, M_{34} - M_{12}\, {M_{23}}^2\, M_{34} - M_{14}\, M_{23}\, {M_{34}}^2,$$
$$-{M_{13}}^2\, {M_{23}}^2 + i\, M_{12}\, M_{14}\, {M_{23}}^2 - i\, M_{14}\, {M_{23}}^2\, M_{34} + i\, {M_{23}}^3\, M_{34}$$
$$-i\, M_{12}\, M_{14}\, {M_{34}}^2 - i\, M_{23}\, {M_{34}}^3,$$

$$M_{12}\,M_{13}\,{M_{23}}^2 + iM_{14}\,{M_{23}}^2\,M_{24} + M_{13}\,M_{14}\,M_{23}\,M_{34},$$

$$-i\,{M_{13}}^2\,M_{14}\,M_{23} + M_{12}\,M_{13}\,M_{14}\,M_{24} - M_{13}\,M_{23}\,M_{24}\,M_{34},$$

$$i\,M_{12}\,M_{13}\,M_{14}\,M_{23} + M_{13}\,M_{14}\,{M_{24}}^2 + i\,M_{13}\,{M_{14}}^2\,M_{34}$$

$$-i\,M_{13}\,M_{14}\,M_{23}\,M_{34} - M_{23}\,M_{24}\,{M_{34}}^2,$$

$${M_{12}}^2\,M_{13}\,M_{24} + M_{13}\,M_{14}\,M_{23}\,M_{24} + i\,{M_{13}}^2\,M_{14}\,M_{34}$$

$$-i\,M_{23}\,{M_{24}}^2\,M_{34} + M_{13}\,M_{24}\,{M_{34}}^2,$$

$$-i\,M_{12}\,M_{13}\,M_{23}\,M_{34} - M_{14}\,M_{23}\,M_{24}\,M_{34} - i\,M_{13}\,M_{14}\,{M_{34}}^2,$$

$$-i\,{M_{13}}^3\,M_{23} + M_{12}\,{M_{13}}^2\,M_{24} - M_{13}\,{M_{23}}^2\,M_{34} - i\,M_{23}\,M_{24}\,{M_{34}}^2 + M_{13}\,{M_{34}}^3,$$

$$i\,M_{12}\,{M_{13}}^2\,M_{23} + M_{13}\,M_{14}\,M_{23}\,M_{24} + i\,{M_{13}}^2\,M_{14}\,M_{34},$$

$${M_{12}}^2\,M_{13}\,M_{23} + M_{13}\,M_{14}\,{M_{23}}^2 + M_{12}\,M_{13}\,M_{14}\,M_{34} - i\,{M_{23}}^2\,M_{24}\,M_{34},$$

$$i\,M_{12}\,M_{13}\,M_{14}\,M_{23} - {M_{12}}^2\,M_{14}\,M_{24} + M_{14}\,{M_{23}}^2\,M_{24} + M_{13}\,M_{23}\,{M_{24}}^2,$$

$$-i\,{M_{12}}^2\,M_{14}\,M_{23} - M_{12}\,M_{14}\,{M_{24}}^2 - i\,M_{12}\,{M_{14}}^2\,M_{34}$$

$$+i\,M_{12}\,M_{14}\,M_{23}\,M_{34} - i\,M_{12}\,{M_{23}}^2\,M_{34} + M_{23}\,{M_{24}}^2\,M_{34} - i\,M_{14}\,M_{23}\,{M_{34}}^2,$$

$$-{M_{12}}^3\,M_{24} - M_{12}\,M_{14}\,M_{23}\,M_{24} + M_{12}\,{M_{23}}^2\,M_{24} + i\,M_{23}\,{M_{24}}^3$$

$$-i\,M_{12}\,M_{13}\,M_{14}\,M_{34} - M_{13}\,{M_{24}}^2\,M_{34},$$

$$i\,M_{12}\,M_{13}\,M_{23}\,M_{24} + M_{14}\,M_{23}\,{M_{24}}^2 + i\,M_{13}\,M_{14}\,M_{24}\,M_{34},$$

$$i\,M_{12}\,{M_{13}}^2\,M_{23} - {M_{12}}^2\,M_{13}\,M_{24} + 2\,M_{13}\,{M_{23}}^2\,M_{24}$$

$$+i\,M_{23}\,{M_{24}}^2\,M_{34} - M_{13}\,M_{24}\,{M_{34}}^2,$$

$$-i\,{M_{12}}^2\,M_{13}\,M_{23} - M_{12}\,M_{14}\,M_{23}\,M_{24} - i\,M_{12}\,M_{13}\,M_{14}\,M_{34},$$

$$-{M_{12}}^3\,M_{23} - M_{12}\,M_{14}\,{M_{23}}^2 + M_{12}\,{M_{23}}^3 + i\,{M_{23}}^2\,{M_{24}}^2$$

$$-{M_{12}}^2\,M_{14}\,M_{34} + M_{14}\,{M_{23}}^2\,M_{34},$$

$$i\,M_{12}\,M_{13}\,{M_{23}}^2 + M_{14}\,{M_{23}}^2\,M_{24} + i\,M_{13}\,M_{14}\,M_{23}\,M_{34},$$

$$-M_{12}\,M_{13}\,M_{14}\,M_{23} + i\,{M_{12}}^2\,M_{14}\,M_{24} + i\,M_{12}\,M_{23}\,M_{24}\,M_{34},$$

$${M_{12}}^2\,M_{13}\,M_{24} - 2\,M_{13}\,{M_{14}}^2\,M_{24} + i\,M_{12}\,M_{14}\,{M_{24}}^2$$

$$+i\,{M_{13}}^2\,M_{14}\,M_{34} + M_{13}\,M_{24}\,{M_{34}}^2,$$

$$-M_{13}\,{M_{14}}^2\,M_{23} + i\,M_{12}\,{M_{14}}^2\,M_{24} + i\,M_{14}\,M_{23}\,M_{24}\,M_{34},$$

$$-{M_{12}}^3\,M_{14} - i\,{M_{13}}^2\,{M_{14}}^2 + M_{12}\,{M_{14}}^3 - M_{12}\,{M_{14}}^2\,M_{23}$$

$$-{M_{12}}^2\,M_{23}\,M_{34} + {M_{14}}^2\,M_{23}\,M_{34},$$

$$-M_{12}\,{M_{13}}^2\,M_{23} + i\,{M_{12}}^2\,M_{14}\,M_{23} + {M_{13}}^2\,M_{14}\,M_{34} + i\,M_{12}\,{M_{14}}^2\,M_{34}$$

$$-i\,M_{12}\,M_{14}\,M_{23}\,M_{34} + i\,M_{12}\,{M_{23}}^2\,M_{34} + i\,M_{14}\,M_{23}\,{M_{34}}^2,$$

$${M_{12}}^2\,M_{13}\,M_{23} - M_{13}\,{M_{14}}^2\,M_{23} - {M_{13}}^2\,M_{14}\,M_{24} + i\,M_{12}\,M_{14}\,M_{23}\,M_{24},$$

$$-{M_{13}}^2\,M_{14}\,M_{23} + i\,M_{12}\,M_{13}\,M_{14}\,M_{24} + i\,M_{13}\,M_{23}\,M_{24}\,M_{34},$$

$$-{M_{12}}^3\,M_{13} - i\,{M_{13}}^3\,M_{14} + M_{12}\,M_{13}\,{M_{14}}^2 - M_{12}\,M_{13}\,M_{14}\,M_{23}$$

$$-{M_{13}}^2\,M_{24}\,M_{34} + i\,M_{12}\,M_{23}\,M_{24}\,M_{34},$$

$${M_{12}}^4 + i\,M_{12}\,{M_{13}}^2\,M_{14} - {M_{12}}^2\,{M_{14}}^2 + {M_{12}}^2\,M_{14}\,M_{23} - {M_{12}}^2\,{M_{23}}^2 + {M_{14}}^2\,{M_{23}}^2$$

$$+{M_{13}}^2\,{M_{24}}^2 - i\,M_{12}\,M_{23}\,{M_{24}}^2.$$

For each point p in the point scheme we now substitute the three linear equations describing $\widetilde{\mathcal{L}}_p$ into the above equations. For example, the lines passing through e_1 are given by the scheme $\mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34})$. Substituting $M_{23} = 0, M_{24} = 0, M_{34} = 0$ into the above quartics we find:

$$M_{12}(M_{12}^3 + iM_{13}^2 M_{14} - M_{12}M_{14}^2)$$

$$-M_{13}(M_{12}^3 + iM_{13}^2 M_{14} - M_{12}M_{14}^2)$$

$$-M_{14}(M_{12}^3 + iM_{13}^2 M_{14} - M_{12}M_{14}^2)$$

Therefore it follows that $\mathcal{L}_{e_1} = \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34}, M_{12}^3 + iM_{13}^2 M_{14} - M_{12}M_{14}^2)$. The computations work exactly the same for the points e_2, e_3, e_4.

The computations for the points $p = [1, a_2, a_3, a_4]$ are slightly more involved. We first substitute $M_{23} = a_2 M_{13} - a_3 M_{12}, M_{24} = a_2 M_{14} - a_4 M_{12}, M_{34} = a_3 M_{14} - a_4 M_{13}$ into the 45 quartics and obtain as one of the equations

$$M_{13}(-a_3 M_{12} + a_2 M_{13}) M_{14}(a_4 M_{12} - a_2 M_{14}) = 0.$$

Thus we have 4 cases to consider: $M_{13} = 0$. $M_{12} = \dfrac{a_2}{a_3} M_{13}$. $M_{14} = 0$. $M_{12} = \dfrac{a_2}{a_4} M_{14}$.

1. $M_{13} = 0$. Then from a computer calculation, we have $M_{12}M_{14}(a_4 M_{12} - a_2 M_{14}) = 0$. The case $M_{12} = 0$ gives the line L_1. The case $M_{14} = 0$ gives no solutions. And finally $M_{12} = \dfrac{a_2}{a_4} M_{14}$ also gives no solutions. We note that $M_{12} = 0$ yields a solution precisely because the point p satisfies $ia_2^2 - a_3 + a_3^3 = 0$.

2. $M_{12} = \dfrac{a_2}{a_3} M_{13}$. We have $M_{13}M_{14}(a_4 M_{13} - a_2 M_{14}) = 0$. The case $M_{13} = 0$ gives

the line L_1 found in Case 1. If $M_{14} = 0$ then since $p = [1, a_2, a_3, a_4]$ satisfies $a_2^2 + a_3^2 a_4^2 = 0$ we get the line L_2. Finally the case $M_{13} = \frac{a_2}{a_4} M_{14}$ has no solutions.

3. $M_{14} = 0$. We immediately have $M_{12}M_{13}(a_3M_{12} - a_2M_{13}) = 0$. If $M_{12} = 0$ then since $p = [1, a_2, a_3, a_4]$ satisfies $a_2 + ia_2^2a_4 - ia_4^3 = 0$ we find the solution L_3. If $M_{13} = 0$ then there are no solutions. While if $M_{12} = \frac{a_2}{a_3} M_{13}$ then we have the solution L_2.

4. $M_{12} = \frac{a_2}{a_4} M_{14}$. We find that $M_{13}M_{14}(a_4M_{13} - a_3M_{14}) = 0$. When $M_{13} = 0$ we have no solutions. If $M_{14} = 0$ then we have the line L_3 and finally when $M_{13} = \frac{a_3}{a_4} M_{14}$ there are no solutions.

   Thus we have proved that $\mathcal{L}_p$ consists of the three lines L_1, L_2, L_3 when $p = [1, a_2, a_3, a_4]$.

It remains to determine the multiplicities of the 3 lines, L_1, L_2, L_3 passing through $p = [1, a_2, a_3, a_4]$. First consider L_1. Let $\mathcal{U}_{ij} \subset \mathbb{P}^5$ denote the affine open set where $M_{ij} \neq 0$. Notice that $L_1 \in \mathcal{U}_{14}$ but L_2, L_3 are not in $\mathcal{U}_{14}$. To find $\mathrm{mult}(L_1)$ we need to calculate the dimension of the local ring $R_{\mathfrak{m}}$ where R is the homogeneous coordinate ring of $\mathcal{L}_p$ and $\mathfrak{m}$ is the maximal graded prime ideal of R corresponding to L_1. Let $\hat{R}$ denote the ring obtained from R by setting $M_{14} = 1$ and let $\hat{\mathfrak{m}}$ be the image of $\mathfrak{m}$ in $\hat{R}$. Then $\mathrm{mult}(L_1) = \dim \hat{R}_{\hat{m}}$.

In $\hat{R}$ we have the relation

$$(a_2 - a_4M_{12})M_{13}(-a_3M_{12} + a_2M_{13}).$$

Since $a_2 - a_4M_{12}$ is not in $\hat{\mathfrak{m}}$ after localizing at $\hat{\mathfrak{m}}$ we have the relation $a_3M_{12}M_{13} - a_2M_{13}^2$. Direct substitution of $M_{13}^2 = \frac{a_3}{a_2}M_{12}M_{13}$ into the relations of $\hat{R}$ gives the relation

$$(a_2 - a_4M_{12})(-a_2a_3M_{12} + a_2^2M_{13} + ia_3M_{13}).$$

Again since $a_2 - a_4M_{12}$ is not in $\hat{\mathfrak{m}}$, after localizing at $\hat{\mathfrak{m}}$, we have the relation $-a_2a_3M_{12}+a_2^2M_{13}+ia_3M_{13}$. We substitute $M_{12} = \frac{M_{13}(a_2^2 + ia_3)}{a_2a_3}$ into the relations of $\hat{R}$ and find the relation $M_{13}^2(-a_3a_4 + a_2^2M_{13} + ia_3M_{13})$. Now $-a_3a_4 + a_2^2M_{13} + ia_3M_{13}$ isn't in $\hat{\mathfrak{m}}$ so we have $M_{13}^2 = 0$ in the local ring $\hat{R}_{\hat{m}}$. Finally we have the relation

$$M_{13}(a_3 - a_4M_{13})(2ia_2^2a_3 - 2ia_2^2a_4M_{13} + a_3a_4M_{13})$$

in $\hat{R}/ < M_{13}^2 >$. Since both $a_3 - a_4M_{13}$ and $2ia_2^2a_3 - 2ia_2^2a_4M_{13} + a_3a_4M_{13}$ are not in $\hat{\mathfrak{m}}$ we have $M_{13} = 0$ in $\hat{R}_{\hat{m}}$. Hence we have eliminated all variables in $\hat{R}_{\hat{m}}$. That is $\hat{R}_{\hat{m}} \cong k$ and $\text{mult}(L_1) = 1$.

An entirely symmetric argument (as above) using the affine open neighborhood $\mathcal{U}_{23}$ of L_3 proves that $\text{mult}(L_3) = 1$. Finally we know that $\mathcal{L}_p$ consists of 6 points counted with multiplicity so $\text{mult}(L_2) = 4$ follows.

$\square$

CHAPTER V

GLOBAL DIMENSION 5 AND QUANTUM $\mathbb{P}^4$.

The purpose of this chapter is to explore the geometry of a quantum $\mathbb{P}^4$. Unfortunately much of the linear geometry disappears as is evidenced by the main theorem of this chapter. We have defined the notions of point module and line module. The obvious generalization is the following definition.

Definition V.1.1. Let $d \in \mathbb{N} \cup \{0\}$.

a) A d-linear module is a graded A-module M which is cyclic and has Hilbert series

$$H_M(t) = \frac{1}{(1-t)^{d+1}}.$$

b) Let $r \in N$. A truncated d-linear module of length r is a graded A-module M which is cyclic and has Hilbert series

$$H_M(t) = \sum_{n=0}^{r-1} \binom{n+d}{d} t^n.$$

Notice that given a d-linear module one can "truncate" it to any desired length. Then 0-linear modules are point-modules, 1-linear modules are line modules. We call 2-linear modules plane modules, and 3-linear modules space modules, with similar terminology for their truncated counterparts.

Here is the main theorem.

Theorem V.1.2. *The generic k-algebra on 5 generators and 10 quadratic relations has no truncated point or truncated line modules of length 3.*

Corollary V.1.3. *The generic k-algebra on 5 generators and 10 quadratic relations has no point or line modules.*

Proof. Given a point or line module M we may truncate it to length 3 to get a truncated point or line module of length 3. However V.1.2 implies there are no such modules.

□

Before beginning the proof of V.1.2 we fix notation. Let k be an algebraically closed field with $\operatorname{char} k \neq 2$. Let V denote a 5-dimensional k vector space, and define

$$A = T(V^*)/I$$

where $T(V^*)$ is the tensor algebra on V^* and $I = \bigoplus_{n\geq 2} I_n$ is a homogeneous ideal of $T(V^*)$. It is then clear that A is connected and generated in degree one. If $M = \bigoplus_{n\in Z} M_n$ is a graded k-vector space with finite dimensional summands then its Hilbert series will be denoted $H_M(t) = \sum_n \dim(M_n)t^n$. Unless otherwise stated all modules will be considered right modules.

In [18], Shelton and Vancliff prove that there is a scheme which represents the functor of d-linear A-modules, and a scheme which represents the functor of truncated d-linear A-modules.

The basic idea is the following. Let M be a d-linear module. Write $M = v_0A$ and let $J = \mathrm{Ann}_{T(V^*)}(v_0)$ where we are considering M as a $T(V^*)$-module. Then J is graded and we keep track of the dimensions of the graded pieces of J inside a product of Grassmannians. This determines the correct scheme.

A final piece of notation. Let $\mathbb{G}_n(V)$ denote the Grassmannian scheme of n-dimensional subspaces of V and $\mathbb{G}^n(V)$ denote the scheme of subspaces of V of codimension n. We also will let $\mathbb{G}(k, n)$ denote the Grassmannian of k-planes in an n-dimensional vector space V.

The goal now is to prove the theorem.

Proof. Suppose that L is a truncated line module of length 3. Write $L = v_0A$. We consider L as a $T(V^*)$ module and set $J = \mathrm{Ann}_{T(V^*)}(v_0)$. Then J is $\mathbb{N}$-graded and note that $J_n = T_n$ for $n \geq 3$. Now we consider dimensions:

$\dim V^* = 5$.

$\dim I_2 = 10$.

$\dim J_1 = 5 - 2 = 3$.

$\dim J_2 = 25 - 3 = 22$.

$\dim J_1 \otimes V^* = 15$.

We have $J_2 \supset I_2 + J_1 \otimes V^*$ so it is necessary that

$$\dim(J_1 \otimes V^* \cap I_2) \geq 3.$$

So we can find $Q \in \mathbb{G}_3(I_2)$ and $N \in \mathbb{G}_2(V)$ such that $N^\perp \otimes V^* \supset Q$. Let

$$\Omega = \{Q \in \mathbb{G}_3(V^* \otimes V^*) : \exists N \in \mathbb{G}_2(V) \text{ such that } N^\perp \otimes V^* \supset Q\}.$$

We now compute the dimension of Ω. First we define the incidence relation

$$\Pi = \{(N, Q) \in \mathbb{G}_2(V) \times \mathbb{G}_3(V^* \otimes V^*) : N^\perp \otimes V^* \supset Q\}$$

We have projection maps $\pi_1 : \Pi \to \mathbb{G}_2(V)$ and $\pi_2 : \Pi \to \mathbb{G}_3(V^* \otimes V^*)$. Notice that π_1 is surjective as given $N \in \mathbb{G}_2(V)$ the dimension of $N^\perp \otimes V^*$ is 15. For $N \in \mathbb{G}_2(V)$, we have $\pi_1^{-1}(N) \cong \mathbb{G}_3(N^\perp \otimes V^*)$. By [9] Theorem 11.14 it follows that Π is irreducible with

$$\dim \Pi = \dim \mathbb{G}_2(V) + \dim \mathbb{G}_3(N^\perp \otimes V^*) = 6 + 36 = 42.$$

Now we analyse π_2. It is clear that the image of π_2 is Ω. For a generic Q in $\operatorname{im} \pi_2$, there will exist a tensor λ of rank 3. By the rank of a tensor we mean the minimal number of pure tensors needed to express it. In other words, there is $\lambda \in Q$ such that $\lambda = a \otimes b + c \otimes d + e \otimes f$ with a, c, e linearly independent. In this case $\pi_2^{-1}(Q) \cong \{N\}$ where $N = \operatorname{Span}\{a, c, e\}^\perp$. That is, π_2 is generically one-to-one. Hence $\dim \Omega = \dim \Pi = 42$.

Now define

$$\Phi = \{(H, Q) \in \mathbb{G}_{10}(V^* \otimes V^*) \times \Omega : H \supset Q\}.$$

Consider the projection maps $\pi_1 : \Phi \to \mathbb{G}_{10}(V^* \otimes V^*)$ and $\pi_2 : \Phi \to \Omega$.

For $Q \in \Omega$ we can extend a basis to find $H \in \mathbb{G}_{10}(V^* \otimes V^*)$ such that $H \supset Q$. So π_2 is surjective. We also have:

$$\pi_2^{-1}(Q) \cong \{H \in \mathbb{G}_{10}(V^* \otimes V^*) : H \supset Q\} \cong \mathbb{G}(7, 22).$$

So it follows that

$$\dim \Phi = \dim \Omega + \dim \mathbb{G}(7, 22) = 42 + 105 = 147.$$

So this proves π_1 is not surjective as $\dim \mathbb{G}_{10}(V^* \otimes V^*) = 150$. So this implies that there are 10-dimensional subspaces H of $V^* \otimes V^*$ which contain no elements of Ω. In other words if we construct our algebra A to have relations given by H then such an algebra will have no truncated line modules of length 3. We remark that the set of 10-dimensional subspaces of $V^* \otimes V^*$ affording an algebra with no truncated line modules of length 3 is an open subset of $\mathbb{G}_{10}(V^* \otimes V^*)$ so such algebras are generic.

We now wish to prove that there are algebras with no truncated point modules of length 3.

Let P be a truncated point module of length 3 and write $P = v_0 A$ and $J = \mathrm{Ann}_{T(V^*)}(v_0) = \bigoplus_{n \geq 0} J_n$. Now considering dimensions we have $\dim J_1 = 4$ and $\dim J_2 = 24$. Necessarily $J_2 \supset J_1 \otimes V^* + I_2$ so that $\dim(J_1 \otimes V^* + I_2) \leq 24$ or equivalently $\dim(J_1 \otimes V^* \cap I_2) \geq 6$. Write $J_1 = \mathrm{Span}\{a, b, c, d\}$. Then every element of $J_1 \otimes V^*$ can be written as $a \otimes v_1 + b \otimes v_2 + c \otimes v_3 + d \otimes v_4$ for some $v_i \in V^*$. Therefore there is $N \in \mathbb{P}(V)$ such that $J_1 \otimes V^* \subset N^\perp \otimes V^*$. Thus we are led to consider the following scheme. Let

$$\Omega = \{Q \in \mathbb{G}_6(V^* \otimes V^*) : \exists N \in \mathbb{P}(V) \text{ such that } N^\perp \otimes V^* \supset Q\}.$$

In order to compute the dimension of Ω we introduce the following incidence relation. Let

$$\Pi = \{(N, Q) \in \mathbb{P}(V) \times \mathbb{G}_6(V^* \otimes V^*) : N^\perp \otimes V^* \supset Q\}.$$

We analyse the canonical projection maps $\pi_1 : \Pi \to \mathbb{P}(V)$ and $\pi_2 : \Pi \to \mathbb{G}_6(V^* \otimes V^*)$. First consider π_1. It is clear that π_1 is onto for if $N \in \mathbb{P}(V)$, we may choose a 6-dimensional subspace Q of $N^\perp \otimes V^*$. Then $(N, Q) \in \Pi$ and $\pi_1(N, Q) = N$. As for the fiber over $N \in \mathbb{P}(V)$ we have

$$\pi_1^{-1}(N) \cong \mathbb{G}_6(N^\perp \otimes V^*).$$

So by [9] Theorem 11.14, $\dim \Pi = \dim \mathbb{P}(V) + \dim \mathbb{G}_6(N^\perp \otimes V^*) = 4 + 84 = 88$.

Now consider π_2. It is clear that $\operatorname{im} \pi_2 = \Omega$. For a generic Q in $\operatorname{im} \pi_2$ there will exist a tensor λ of rank 4. In other words there is $\lambda \in Q$ such that $\lambda = a \otimes b + c \otimes d + e \otimes f + g \otimes h$ with a, c, e, $g \in V^*$ linearly independent. In this case $\pi_2^{-1}(Q) \cong \{N\}$ where $N = \text{Span}\{a, c, e, g\}^\perp$. That is, π_2 is generically one-to-one. Hence $\dim \Omega = \dim \Pi = 88$.

Now define

$$\Phi = \{(L, Q) \in \mathbb{G}_{10}(V^* \otimes V^*) \times \Omega : L \supset Q\}.$$

Consider the projection maps $\pi_1 : \Phi \to \mathbb{G}_{10}(V^* \otimes V^*)$ and $\pi_2 : \Phi \to \Omega$.

For $Q \in \Omega$ we can extend a basis to find $L \in \mathbb{G}_{10}(V^* \otimes V^*)$ such that $L \supset Q$. So π_2 is surjective. We also have:

$$\pi_2^{-1}(Q) \cong \{L \in \mathbb{G}_{10}(V^* \otimes V^*) : L \supset Q\} \cong \mathbb{G}_4(V^* \otimes V^*/Q).$$

So it follows that

$$\dim \Phi = \dim \Omega + \dim \mathbb{G}_4(V^* \otimes V^*/Q) = 88 + 60 = 148.$$

So this proves π_1 is not surjective as $\dim \mathbb{G}_{10}(V^* \otimes V^*) = 150$. So this implies that there are 10-dimensional subspaces L of $V^* \otimes V^*$ which contain no elements of Ω. In other words if we construct our algebra A to have relations given by L then such an algebra will have no truncated point modules of length 3. We remark that the set of such 10-dimensional subspaces is open in $\mathbb{G}_{10}(V^* \otimes V^*)$ so that algebras with no truncated point modules of length 3 are generic. This finishes the proof.

□

We now wish to explore the geometry of the plane and space modules of a quantum $\mathbb{P}^4$. We will describe the scheme of plane modules and the scheme of space modules.

Assume from now on that A has the same Hilbert series as the commutative polynomial ring on 5 variables, i.e.,

$$H_A(t) = \frac{1}{(1-t)^5}$$

and that A is a domain.

A description of the right 3-linear modules, or "space"-modules is straightforward. Let S be a space-module and write $S = v_S A$ where $v_S \in S_0 = kv_S$. Now $\dim A_1 = 5$ and $\dim S_1 = 4$ so it follows that $\operatorname{Ann}_{A_1}(S_0) = ku$ for some $u \in A_1$. Now $A/uA \twoheadrightarrow S$ and since A/uA and S have the same Hilbert series they must be isomorphic. Hence the scheme of right space modules can be identified with $\mathbb{P}(A_1) = \mathbb{P}(V^*) \cong \mathbb{P}^4$.

We now seek a description of the right plane modules. Consider

$$\Omega_2(A,2) \subset \Upsilon_2(V,2) = \mathbb{G}^3(V^*) \times \mathbb{G}^6(V^* \otimes V^*)$$

Let $P^\perp = \operatorname{im}(\bar{\pi}_1 : \Omega_2(A,2) \to \mathbb{G}^3(V^*))$. For $Q \in \mathbb{G}^3(V^*)$ we have $Q \in P^\perp$ if and only if there is $Q_1 \in \mathbb{G}^6(V^* \otimes V^*)$ such that $I_2 + Q \otimes V^* \subset Q_1$. So it is necessary that $\dim(I_2 + Q \otimes V^*) \leq 19$ or equivalently

$$\dim(I_2 \cap Q \otimes V^*) \geq 1.$$

Hence

$$P^\perp = \{Q \in \mathbb{G}^3(V^*) : \dim(I_2 \cap Q \otimes V^*) \geq 1\}.$$

We now want to take $Q \in P^\perp$ and construct in a canonical way a plane module. So let $Q \in P^\perp$ and choose a basis $\{v, w\} \subset V^*$ for Q. Set

$$M(Q) := A/QA = A/(vA + wA).$$

Choose $a, b \in V^*$ so that $0 \neq v \otimes a + w \otimes b \in I_2$. This is possible as $\dim(I_2 \cap Q \otimes V^*) \geq 1$. Since A is a domain, neither a nor b is 0. Let S be the space module A/wA. Let $\bar{v} = v + wA \in S$ so that $\bar{v}A$ is a submodule of S. If we assume S is graded homogeneous with respect to GKdim then $\operatorname{GKdim}(\bar{v}A) = 4$. Notice that $\bar{v}a = 0$ in S so the canonical map $(A/aA)[-1] \twoheadrightarrow \bar{v}A$ is well-defined. If this epimorphism had a kernel then $\operatorname{GKdim}(\bar{v}A) = 3$, a contradiction. Hence $(A/aA)[-1] \cong \bar{v}A$. So $H_{\bar{v}A} = \dfrac{t}{(1-t)^4}$. We have $M(Q) \cong S/\bar{v}A$ so

$$H_{M(Q)} = H_S - H_{\bar{v}A} = \frac{1}{(1-t)^4} - \frac{t}{(1-t)^4} = \frac{1}{(1-t)^3}.$$

So $M(Q)$ is a plane module.

Now let $M(r)$ denote any truncated plane module of length $r+1$. Let $Q = \mathrm{Ann}_{A_1}(v_L)$ where $Q = v_L A$ and $\deg(v_L) = 0$. Since $\dim Q = \dim A_1 - \dim M(r)_1 = 5 - 3 = 2$ and since $\dim(Q \otimes V^* + I_2) \leq \dim T(V^*)_2 - \dim M(r)_2 = 25 - 6 = 19$ we have $\dim(Q \otimes V^* \cap I_2) \geq 1$. Therefore $Q \in P^\perp$ and $M(Q)$ is a plane module. Note that $M(r)$ is a quotient of $M(Q)$ and by comparing Hilbert series $M(r) \cong M(Q)(r)$.

Another useful way of describing the plane scheme is inside the projective space of the relations $\mathbb{P}(I_2) \cong \mathbb{P}^9$. Note that plane modules are essentially determined by 2-tensors so we will make the following identification.

Given a pure tensor $a \otimes b \in V^* \otimes V^*$, we may consider it as an element of $\mathrm{Hom}_k(V, V^*)$ via: for $v \in V$ let $(a \otimes b)(v) = a(v).b$. Extending linearly we make the following identification

$$\mathbb{P}(V^* \otimes V^*) \cong \mathbb{P}(\mathrm{Hom}_k(V, V^*)) \cong \mathbb{P}^{24}.$$

We may now unambiguously refer to the rank of a 2-tensor λ. Let

$$G = \mathbb{P}(I_2) \cap \{\lambda \in \mathbb{P}(V^* \otimes V^*) : \mathrm{rank}(\lambda) = 2\}$$

and let $\bar{G}$ be its scheme-theoretic closure in $\mathbb{P}(V^* \otimes V^*)$.

Proposition V.1.4. *$G = \bar{G}$ and $G \cong P^\perp$.*

Proof. Define a map $\psi : G \to \mathbb{G}^3(V^*)$ by $\psi(\lambda) = \ker(\lambda)^\perp$ for $\lambda \in G$.

Consider the incidence relation

$$\Psi = \{(Q, \lambda) \in \mathbb{G}^3(V^*) \times \mathbb{P}(V^* \otimes V^*) : \lambda \subset Q \otimes V^* \cap I_2\}.$$

Let $\pi_1 : \Psi \to \mathbb{G}^3(V^*)$ and $\pi_2 : \Psi \to \mathbb{P}(V^* \otimes V^*)$ be the restrictions of the canonical projections. For $Q \in \mathbb{G}^3(V^*)$ we have

$$\pi_1^{-1}(Q) = \{(Q, \lambda) : \lambda \subset Q \otimes V^* \cap I_2\} \cong \mathbb{P}(Q \otimes V^* \cap I_2).$$

For $\lambda \in \mathbb{P}(V^* \otimes V^*)$ we consider cases for the fiber

$$\pi_2^{-1}(\lambda) = \{(Q, \lambda) : \lambda \subset Q \otimes V^* \cap I_2\}.$$

If $\text{rank}(\lambda) \geq 3$ or $\lambda \notin \mathbb{P}(I_2)$ then $\pi_2^{-1}(\lambda) = \emptyset$. If $\text{rank}(\lambda) = 1$ and $\lambda \in \mathbb{P}(I_2)$ then $\lambda = a \otimes b$ for some a , b in $\mathbb{P}(V^*)$ and then

$$\pi_2^{-1}(\lambda) = \{(Q, \lambda) : a \subset Q\} \cong \mathbb{P}^3.$$

If $\text{rank}(\lambda) = 2$ and $\lambda \in \mathbb{P}(I_2)$ then

$$\pi_2^{-1}(\lambda) = \{(Q, \lambda) : \lambda \subset Q \otimes V^*\}.$$

If $Q \in \mathbb{G}^3(V^*)$ and $\lambda \in Q \otimes V^*$ write $\lambda = a \otimes b + c \otimes d$ where a, c span Q and b, d are linearly independent. Then $v \in \ker \lambda$ if and only if $a(v).b + c(v).d = 0$ if and only if $a(v) = c(v) = 0$ which is equivalent to $v \in Q^\perp$. So this proves

$$\{(Q, \lambda) : \lambda \subset Q \otimes V^*\} \subset \{(Q, \lambda) : Q^\perp = \ker \lambda\}.$$

Conversely let $Q \in \mathbb{G}^3(V^*)$ and suppose $Q^\perp = \ker \lambda$. Write $\lambda = a \otimes b + c \otimes d$ where a, c and b, d are linearly independent, which we may do as the rank of λ is 2. Then it follows, as in the above calculation that for $v \in \ker \lambda$, $a(v) = c(v) = 0$, so that a and c are in Q. This implies $\lambda \subset Q \otimes V^*$. Hence

$$\{(Q, \lambda) : \lambda \subset Q \otimes V^*\} = \{(Q, \lambda) : Q^\perp = \ker \lambda\} \cong \mathbb{P}^0.$$

Since we are assuming A is a domain, no element of I_2 has rank one. So we have $G = \bar{G}$. Furthermore, π_2 is injective on closed points. So by [18] Lemma 2.4, π_1, is also injective on closed points. By [18] Lemma 1.6, it follows that π_1 and π_2 are closed immersions. Clearly the image of π_1 is $P^\perp$ and the image of π_2 is G, so that Ψ is the graph of $\varphi : \mathbb{G}^3(V^*) \to G$ where $\varphi(Q) = Q \otimes V^* \cap I_2$. Then $G \cong P^\perp$ as φ is the inverse to ψ. This finishes the proof.

□

The subscheme of tensors of $\mathbb{P}(V^* \otimes V^*)$ of rank at most 2 has dimension 15 while the dimension of $\mathbb{P}(I_2)$ has dimension 9. So it follows that the irreducible components of the plane scheme have dimension at least 0. So in this case plane modules always exist and generically we expect there to be only finitely many of them. The degree of the subscheme of tensors of $\mathbb{P}(V^* \otimes V^*)$ of rank at most 2 is, by Example 19.10 [9], 175. Then Bezout's theorem implies: generically there are 175 plane modules counted with multiplicity.

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