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CHAPTER I

THE ETA INVARIANT

I.1 Introduction

In the first chapter, we discuss the basic analytic underpinnings of the theory with which we will work. In the following sections, we review the theory of elliptic partial differential operators which we shall need. We will discuss Sobolev spaces, spectral theory, and the eta invariant.

I.2 Partial Differential Operators

We start with a description of partial differential operators on $\mathbb{R}^m$ and their symbols. We use this later in order to build partial differential operators on manifolds. Let $x := (x^1, ..., x^m)$ be the standard coordinates on $\mathbb{R}^m$, let $\alpha := (\alpha_1, ..., \alpha_m)$ for $\alpha_i \in \mathbb{N}$ be a multi-index. Set $|\alpha| := \alpha_1 + ... + \alpha_m$. Let $C_0^\infty(\mathbb{R}^m)$ be the set of smooth functions in $\mathbb{R}^m$ with compact support. The partial differential operator $D_x^\alpha : C_0^\infty(\mathbb{R}^m) \to C_0^\infty(\mathbb{R}^m)$ is defined by

$$D_x^\alpha := (\sqrt{-1})^{|\alpha|}(\partial_1^x)^{\alpha_1}...(\partial_m^x)^{\alpha_m}.$$

If $a_\alpha \in C_0^\infty(\mathbb{R}^m)$,

$$P := \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha$$

is a partial differential operator of order d. We define the symbol of P by replacing D_x^α by the dual variable $\xi^\alpha := \xi_1^{\alpha_1}...\xi_m^{\alpha_m}$. Thus

$$\sigma(P) = p(x,\xi) := \sum_{|\alpha|\leq d} a_\alpha(x)\xi^\alpha.$$

An analytical representation of the operator P can be given as follows. The Fourier transform of $f \in C_0^\infty(\mathbb{R}^m)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^m} e^{-\sqrt{-1}x\cdot\xi} f(x)dx.$$

Now we can express P in terms of the symbol $\sigma(P) = p$ by

$$Pf(x) = (4\pi)^{-m/2} \int_{\mathbb{R}^m} e^{\sqrt{-1}x\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi,$$

see Gilkey [18, Lemma 2.1.1] for details. The leading symbol is the polynomial of order d in the dual variable ξ defined by

$$\sigma_L(P) = p_d(x,\xi) := \sum_{|\alpha|=d} a_\alpha(x)\xi^\alpha.$$

I.1 Example. The geometer's Laplacian on $\mathbb{R}^m$ is defined by $\Delta f := -\sum_{i=1}^m (\partial_i^x)^2$. The symbol of Δ agrees with the leading symbol in this case;

$$\sigma(\Delta) = \sigma_L(\Delta) = |\xi|^2 = \xi_1^2 + ... + \xi_m^2.$$

This will play a crucial role in what follows.

I.3 Sobolev Spaces on R^m

We wish to regard our operators as acting on Sobolev spaces. With this, we will be able to use the analytical framework of Hilbert space theory. We define these spaces as follows. For any $s \in \mathbb{R}$ and for any $f \in C_0^\infty(\mathbb{R}^m)$, define the Sobolev norms by

$$||f||_s^2 := \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

The Sobolev space $H_s(\mathbb{R}^m)$ is the Hilbert space completion of $C_0^\infty(\mathbb{R}^m)$ with respect to this norm. Note that $H_0(\mathbb{R}^m) = L^2(\mathbb{R}^m)$. More generally, $H_s(\mathbb{R}^m)$ is isomorphic L^2 with the weight function $(1+|\xi|^2)^{s/2}$. If s is a non-negative integer, an equivalent norm for $H_s(\mathbb{R}^m)$ is given by

$$||f||_s^2 = \sum_{|\alpha|\le s} \int |D_x^\alpha f|^2 dx.$$

Hence, we can think of the subscript s as measuring the degree of smoothness; s should be thought of measuring the number of L^2 derivatives.

If P is a partial differential operator of order d, the estimate

$$||Pf||_{s-d} \le C(P,s)||f||_s$$

shows that P extends to a continuous operator

$$P : H_s(\mathbb{R}^m) \to H_{s-d}(\mathbb{R}^m)$$

on Sobolev spaces. We refer to Gilkey [18, Lemma 1.1.3].

We will also use the sup norm to measure derivatives. For k in $\mathbb{N}$ and for f in $C_0^\infty(\mathbb{R}^m)$, we define

$$||f||_{\infty,k} = \sup_{x\in\mathbb{R}^m} \sum_{|\alpha|\le k} |D_x^\alpha f(x)|.$$

The completion of $C_0^\infty(\mathbb{R}^m)$ with respect to this norm is a subset of $C^k(\mathbb{R}^m)$.

I.4 Sobolev Spaces and Operators on Manifolds

Now we will extend these definitions to manifolds. Unless otherwise stated, all manifolds are assumed to be compact, smooth, Riemannian manifolds, without boundary. Let $C^\infty(M)$ be the space of smooth functions on M. Let $\{U_i, h_i\}$ be a coordinate atlas for M. The collection of open sets $\{U_i\}$ forms an open cover for M and the local diffeomorphisms $h_i : U_i \to h(U_i) \subset \mathbb{R}^m$ give local coordinates on M. Let ϕ_i be a partition of unity subordinate to this atlas. Let $(h_i)_*$ mapping $C^\infty(U_i)$ to $C^\infty(h(U_i))$ be the push forward. The definition of the norm on the Sobolev spaces $H_s(M)$ is

$$||f||_s^2 := \sum_i ||(h_i)_*(\phi_i f)||_s^2;$$

the norm depends on the choices made. However, changing the atlas or the partition of unity gives an equivalent norm so the spaces $H_s(M)$ are invariantly defined as Banach spaces.

Now we extend this theory to vector bundles over a manifold. All vector bundles are assumed to be smooth. Let $C^\infty(V)$ be the space of smooth sections of a smooth vector bundle V over M. We choose a complementary vector bundle W so that $V \oplus W = 1^k$ for some k. This defines an embedding $\mathcal{J}$ of $C^\infty(V)$ in $C^\infty(1^k)$. If $f \in C^\infty(V)$, let $\mathcal{J}(f) = (\mathcal{J}_1(f), ..., \mathcal{J}_k(f))$ where $\mathcal{J}_i(f) \in C^\infty(M)$. We define

$$||f||_s^2 := \sum_i ||\mathcal{J}_i f||_s^2$$

and let $H_s(V)$ be the completion of $C^\infty(V)$ with respect to this norm. This depends on the choice of $\mathcal{J}$. However, changing $\mathcal{J}$ gives an equivalent norm so the spaces $H_s(V)$ are once again invariantly defined as Banach spaces.

We say that P is a partial differential operator of order d from $C^\infty(V_1)$ to $C^\infty(V_2)$ if P is given by a matrix of partial differential operators of order d in any coordinate chart over which the V_i for $i = 1, 2$ are trivialized. We can extend such an operator to a continuous map

$$P : H_s(V_1) \to H_{s-d}(V_2).$$

The leading symbol of P is invariantly defined. If $T^*(M)$ is the cotangent space of M, then

$$\sigma_L(P) : T^*M \to \mathrm{End}(V_1, V_2).$$

If $\phi \in C_0^\infty(\mathbb{R}^m)$ and if $f \in C^\infty(V_1)$, then

$$\sigma_L(P)(d\phi)f = \lim_{t\to\infty} t^{-d} e^{\sqrt{-1}t\phi} P(e^{-\sqrt{-1}t\phi} f).$$

This will play a crucial role in what follows.

I.5 Elliptic Operators

A partial differential operator P of order d on $C^\infty(V)$ is called *elliptic* if the leading symbol is invertible for $\xi \neq 0$; i.e.

$$\det(p_d(x, \xi)) \neq 0 \text{ for } \xi \neq 0.$$

A second order D is said to be of Laplace type if the leading symbol of D is given by the metric tensor. This means that locally D has the form

$$D = -g^{ij}\partial_i\partial_j - A^k\partial_k - B$$

where A^k and B are endomorphisms of V and where the inverse of the metric tensor g^{ij} acts by scalar multication; we adopt the *Einstein convention* and sum over repeated indices. We say that an operator P is of *Dirac type* if P^2 is of Laplace type. Operators of Laplace type and of Dirac type are elliptic. We refer to §II.3 for further details.

I.6 Connections

A connection ∇ is a first order partial differential operator

$$\nabla : C^\infty(V) \to C^\infty(T^*M \otimes V)$$

which satisfies the Leibnitz rule

$$\nabla(\phi f) = \phi \nabla f + d\phi \otimes f \ \forall \phi \in C^\infty(M), \forall f \in C^\infty(V).$$

Note that $\sigma_L(\nabla)(x,\xi)(v) = \sqrt{-1}\xi \otimes v$.

I.7 Spectral Theory

Let $dvol$ be the Riemannian volume element on M. We suppose that V is equipped with a smooth pointwise fiber metric over M and let

$$(f,g)_{L^2(V)} := \int_M (f,g) dvol.$$

The following result is well known, see for example Gilkey [18, Lemma 1.6.3].

1.2 Lemma. *Let P be a self-adjoint, elliptic, partial differential operator of order $d > 0$.*

(1) *There exists a complete orthonormal basis ϕ_n for $L^2(V)$ of smooth sections ϕ_n so that $P\phi_n = \lambda_n \phi_n$.*

(2) *For any k, there exists $\ell(k)$ such that $||\phi_n||_{k,\infty} \le C(k,P)(1 + |\lambda_n|)^{\ell(k)}$.*

(3) *We can order the eigenvalues so $|\lambda_1| \le |\lambda_2| \le \ldots$. There exists $\epsilon(d) > 0$ so that $|\lambda_n| \ge C(P)n^{\epsilon(d)}$ if $n \ge n_0$ is large enough.*

(4) *Let $c_n(f) = (f, \phi_n)_{L^2}$ be the Fourier coefficients. For every j, there exists $k(j)$ such that if $f \in C^{k(j)}(V)$, then $\sum_n |c_n(f)\lambda_n^j| < \infty$. Thus the series $\sum_n c_n(f)\phi_n$ converges uniformly to f in the C^j norm.*

The collection $\{\phi_n, \lambda_n\}$ is called a *discrete spectral resolution* of P. Let

$$E(P, \lambda) := \{f \in C^\infty(V) : Pf = \lambda f\}$$

be the eigenspaces of P. Lemma 1.2 implies that $\dim E(P, \lambda) < \infty$ and there is an orthogonal direct sum:

$$L^2(V) = \oplus_\lambda E(P, \lambda).$$

This will play a crucial role in what follows.

I.8 The Eta Invariant

Let P be a self-adjoint operator of Dirac type. We define

$$\begin{aligned} \eta(s, P) :=& \sum_n \operatorname{sign}(\lambda_n)|\lambda_n|^{-s} + \dim\ker(P) \\ =& \sum_\lambda \dim(E(P,\lambda))\operatorname{sign}(\lambda)|\lambda|^{-s} + \dim\ker(P). \end{aligned} \tag{1.5}$$

By Lemma 1.2, $|\lambda_n| \geq n^{\epsilon(d)}$ for n large. Thus the series in equation (1.5) converges absolutely to a holomorphic function of s if $\mathrm{Re}(s) >> 0$.

A global invariant $A(P)$ is said to be *locally computable* if there exists a local invariant $\mathcal{A}(P)(x)$ so that $A(P) = \int_M \mathcal{A}(P) dvol$.

1.3 Theorem. *Let P be a self-adjoint operator of Dirac type on a manifold M of dimension m.*

(1) *The function $\eta(s, P)$ has a meromorphic extension to $\mathbb{C}$. The poles of η are simple and located $s = m - n$ for $n \in \mathbb{N}$. The residue of η at such a pole is locally computable.*

(2) *The function $\eta(s, P)$ is regular at $s = 0$.*

Proof. One can use the calculus of pseudo differential operators depending on a complex parameter, which was developed by Seeley [35, 36], to prove the first assertion of the theorem; the second follows from the work of Atiyah et al. [2, 3, 4]. See Gilkey [19, §1.2] for further details. $\square$

Let N be a compact manifold with smooth boundary M. Let

$$D : C^\infty(V_1) \to C^\infty(V_2)$$

define an elliptic complex over N. We assume the structures are product near the boundary M and the complex is of Dirac type. This means near M that D takes the form:

$$D = \sigma(\partial_\nu + P)$$

where P is a self-adjoint operator of Dirac type on $C^\infty(V_1)$ and where σ is a unitary isomorphism between V_1 and V_2 which is given by the leading symbol of D applied to the inward unit normal vector ∂_ν. Impose spectral boundary conditions; see [2] for details. Let $\mathcal{P}$ be the integrand of the local index theorem; $\mathcal{P}$ vanishes if m is even and is locally computable if m is odd. The following is the extension of the index theorem to manifolds with boundary proved by Atiyah, Patodi, and Singer [2, 3, 4].

1.4 Theorem. $\mathrm{Index}(D) = \int_N \mathcal{P} - \eta(P,s)/2|_{s=0}$.

Let

$$\eta(P) := \eta(s,P)/2|_{s=0} \in \mathbb{R}/\mathbb{Z} \tag{1.6}$$

be a measure of the spectral asymmetry of the operator P. It will play a crucial role in what follows.

1.7 Theorem. *If $P(\varepsilon)$ is a smooth 1-parameter family family of operators of Dirac type, then $\eta(P(\varepsilon))$ is a smooth function of ε. The derivative $\eta_\varepsilon(P(\varepsilon))$ is locally computable. If m is even, $\eta(P(\varepsilon))$ is independent of ε.*

Proof. This follows from Theorem 1.3 and from the Seeley calculus; see Gilkey [19, §1.2] for further details.

I.8 Example. Let $P := -\sqrt{-1}\partial_\theta$ on $C^\infty(S^1)$. Let $P(\varepsilon) := P - \varepsilon$. Then we have $\{n-\varepsilon, e^{\sqrt{-1}n\theta}\}$ for $n \in \mathbb{Z}$ is the spectral resolution of $P(\varepsilon)$. Thus

$$\eta(s, P(\varepsilon)) = \sum_{n\in\mathbb{Z}} \operatorname{sign}(n-\varepsilon)|n-\varepsilon|^{-s} + \dim\ker(P(\varepsilon)).$$

The spectrum of $P(0)$ is symmetric with respect to the origin so $\eta(s, P(0)) = 1$. Thus $\eta(P(0)) = 1/2$ in $\mathbb{R}/\mathbb{Z}$. For $0 < |\varepsilon| < 1$, define

$$\zeta(s,\varepsilon) := \sum_{n\geq 0}(n+\varepsilon)^{-s}.$$

This has a meromorphic extension to $\mathbb{C}$ with an isolated simple pole at $s = 1$; the residue at $s = 1$ is $+1$. We compute:

$$\begin{aligned}\eta(s, P(\varepsilon)) =& \sum_{n\geq 1}(n-\varepsilon)^{-s} - \sum_{n\geq 0}(n+\varepsilon)^{-s}\\ =& \zeta(s, 1-\varepsilon) - \zeta(s,\varepsilon).\end{aligned}$$

This is an entire function of s since the poles at $s = 1$ cancel. We differentiate with respect to the parameter ε to see that

$$\dot\eta(s, P(\varepsilon)) = s\sum_{n\in\mathbb{Z}}|n-\varepsilon|^{-s-1} = s\{\zeta(s+1, 1-\varepsilon) + \zeta(s+1,\varepsilon)\}. \tag{1.9}$$

Since $\zeta(s+1, 1-\varepsilon)$ and $\zeta(s+1,\varepsilon)$ have simple poles at $s = 0$ with residue $+1$, we have that $\{s\zeta(s+1, 1-\varepsilon)\}|_{s=0} = 1$ and $\{s\zeta(s+1,\varepsilon)\}|_{s=0} = 1$. Therefore equation (1.9) implies

$$\dot\eta(P(\varepsilon)) = \dot\eta(s, P(\varepsilon))/2|_{s=0} = 1 \text{ for } 0 < |\epsilon| < 1.$$

By continuity, $\dot\eta(P(0)) = 1$. Since $\eta(P(0)) = 1/2$, we see $\eta(P(\varepsilon)) = 1/2 + \varepsilon$ in $\mathbb{R}/\mathbb{Z}$. Since the spectrum of $P(\varepsilon)$ is periodic with period 1, we must reduce mod $\mathbb{Z}$ to ensure that $\eta(P)$ is continuous with respect to one parameter families.

CHAPTER II

EQUIVARIANT BORDISM AND CONNECTIVE K THEORY

II.1 Introduction

In this chapter, we define Clifford algebras, *spin* and *pin* structures, and the Dirac operator. We define the equivariant bordism groups we shall need and extend the eta invariant to this setting. We postpone until a later chapter a discussion of the twisted bordism groups.

II.2 Clifford Algebras

Let $Clif^{\pm}(m)$ denote the real Clifford algebra on $\mathbb{R}^m$. This is the universal unital algebra generated by $\mathbb{R}^m$ subject to the Clifford commutation relations

$$v * w + w * v = \pm(v, w)1.$$

Let $Clif^c(m) := Clif^-(m) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. Note $Clif^-(m) \otimes_{\mathbb{R}} \mathbb{C}$ and $Clif^+(m) \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic.

II.3 Operators of Dirac Type

In §I.5, we said that a first order partial differential operator P was of *Dirac type* if P^2 is of Laplace type; i.e. the leading symbol of P^2 is given by the metric tensor.

For such an operator, the normalized leading symbol $\gamma := -\sqrt{-1}p$ of P gives a vector bundle V over M a $Clif^{-}(T^*M)$ module structure i.e. $\gamma : T^*M \to End(V)$ satisfies $\gamma(\xi)^2 = -|\xi|^2 I_V$. Conversely, given a $Clif^{-}(T^*M)$ module structure γ on a vector bundle V and a connection ∇ on V, then $P = P(\gamma, \nabla) := \gamma \circ \nabla$ is an operator of Dirac type on V. Choose a fiber metric on V so that γ is skew-adjoint; such metrics always exist. A connection ∇ is said to be compatible if ∇ is Riemannian and if $\nabla p = 0$. Such connections always exist, see Branson and Gilkey [14]. If ∇ is compatible, then $P(\gamma, \nabla)$ is self-adjoint.

II.4 Pinor Groups

Let $Pin^{\pm}(m) \subset Clif^{\pm}(m)$ be the multiplicative subgroup generated by the unit sphere of $\mathbb{R}^m$; i.e.

$$Pin^{\pm}(m) = \{x = v_1 * ... * v_k : |v_i| = 1 \text{ for some } k\}.$$

Define the following groups and representations

$Pin^c(m) := Pin^{-}(m) \times_{\mathbb{Z}_2} S^1$ where we identify (g, λ) and $(-g, -\lambda)$,

$\det : Pin^c(m) \to S^1$ by $\det(g, \lambda) = \lambda^2$,

$\chi : Pin^{\pm}(m) \to \mathbb{Z}_2$ by $\chi(v_1 * ... * v_k) = (-1)^k$, and

$\Psi : Pin^{\pm}(m) \to O(m)$ by $\Psi(x) : w \mapsto \chi(x) x * w * x^{-1}$.

$Spin(m) = \ker(\chi) \cap Pin^{-}(m) \approx \ker(\chi) \cap Pin^{+}(m)$, and

$Spin^c(m) = Spin(m) \times_{\mathbb{Z}_2} S^1$.

Let $m \geq 3$. Then Ψ defines a surjective group homomorphism from $Spin(m)$ to the orthogonal group $SO(m)$. Since $Spin(m)$ is connected, $\pi_1(SO(m)) = \mathbb{Z}_2$, and $\ker(\Psi) = \{\pm 1\} \subset Spin(m)$, we have $Spin(m)$ is the universal covering group of $SO(m)$.

Note that Ψ defines a surjective group homomorphism from $Pin^{\pm}(m)$ to the orthogonal group $O(m)$; this exhibits $Pin^{\pm}(m)$ a universal covering groups of $O(m)$. Since $O(m)$ is not connected, the universal cover is not uniquely defined as a group; one must decide how to multiply the arc components and $Pin^{\pm}(m)$ are the two possible universal covering groups. We extend χ and Ψ to $Pin^{c}(m)$ by defining $\chi(x, \lambda) = \chi(x)$ and $\Psi(x, \lambda) = \Psi(x)$.

II.5 Pin and Spin Structures

Let V be a real vector bundle of dimension ν with an inner product. We say that V admits a *pin*$^{\pm}$ or a *pin*^{c} structure if we can lift the transition functions of V from the orthogonal group $O(\nu)$ to the group $Pin^{\pm}(\nu)$ or $Pin^{c}(\nu)$. We say that V admits a *spin* or a *spin*^{c} structure if V is orientable and if we can lift the transition functions to $Spin(\nu)$ or $Spin^{c}(\nu)$. We say that a manifold M admits such a structure if the tangent bundle $T(M)$ admits this structure.

This condition can be expressed in terms of characteristic classes. Let $w_i(V)$ for $i = 1, 2$ be the first two Stiefel-Whitney classes of V. We refer to Giambalvo [16] for the proof of the following results. It shows that we can stabilize; a bundle V admits a suitable structure if and only if $V \oplus 1$ admits this structure.

2.1 Lemma.

(1) *The bundle V admits a spin structure* $\iff$ $w_1(V) = 0$ *and* $w_2(V) = 0$.

(2) *The bundle V admits a spin^c structure* $\iff$ $w_1(V) = 0$ *and if* $w_2(V)$ *lifts from* $H^2(M; \mathbb{Z}_2)$ *to* $H^2(M; \mathbb{Z})$.

(3) *The bundle V admits a pin$^-$ structure* $\iff$ $w_1(V)^2 + w_2(V) = 0$.

(4) *The bundle V admits a pin$^+$ structure* $\iff$ $w_2(V) = 0$.

(5) *The bundle V admits a pin^c structure* $\iff$ $w_2(V)$ *lifts from* $H^2(M; \mathbb{Z}_2)$ *to* $H^2(M; \mathbb{Z})$.

II.6 Projective Spaces

Let S^k be the unit sphere in $\mathbb{R}^{k+1}$. Let $\mathbb{RP}^k := S^k/\mathbb{Z}_2$ be real projective space of dimension k where we identify x to $-x$ in S^k. Let L be the classifying line bundle over $\mathbb{RP}^k$. Consider the direct sum of ν copies of L, $\nu L := L \oplus ... \oplus L$. We give a geometrical argument to determine when νL admits suitable structures rather than an argument involving characteristic classes. For $\epsilon = +$, $\epsilon = -$, or $\epsilon = c$, let $\omega_\nu := e_1 * ... * e_\nu \in Pin^\epsilon(\nu)$. Then $\Psi(\omega_\nu) = -I_\nu$ so ω is a lift of the transition functions of νL from $O(\nu)$ to $Pin^\epsilon(\nu)$. Consequently νL admits a $pin^\pm$ structure if and only if $\omega_\nu^2 = 1$ or equivalently if $(\pm 1)^\nu(-1)^{\nu(\nu-1)/2} = 1$; by replacing ω_ν by $\sqrt{-1}\omega_\nu$ if necessary we see νL always admits a pin^c structure. These structures reduce to $spin^\epsilon$ structures if and only if ν is even. Note that $T(\mathbb{RP}^k) \oplus 1 = (k+1)L$. Hence

a) $\mathbb{RP}^{4k}$ and $(4k+1)L$ admit pin^+ structures.

b) $\mathbb{RP}^{4k+1}$ and $(4k+2)L$ admit $spin^c$ structures.

c) $\mathbb{RP}^{4k+2}$ and $(4k+3)L$ admit pin^- structures.

d) $\mathbb{RP}^{4k+3}$ and $(4k+4)L$ admit $spin$ structures.

Note: we have $w(\nu L) = (1+x)^\nu$ where x generates $H^1(\mathbb{RP}^m; \mathbb{Z}_2) = \mathbb{Z}_2$. Thus $w_1(\nu L) = \nu x$ and $w_2(\nu L) = \nu(\nu-1)/2$. We can now use Lemma 2.1 to derive these relations rather than the geometric argument given above.

II.7 Representations of Finite Groups

Let π be a finite group. Let $Irr(\pi)$ be the set of equivalence classes of irreducible unitary representations of π; $|Irr(\pi)|$ is the number of conjugacy classes of π. Let

$$RU(\pi) := \oplus_{\rho \in Irr(\pi)} \rho \cdot \mathbb{Z}$$

be the unitary group representation ring of π; $RO(\pi)$ and $RSp(\pi)$ is similar. Tensor product defines the ring structure on $RU(\pi)$ and $RO(\pi)$; $RSp(\pi)$ is an Abelian group because the tensor product of two symplectic representations need not be symplectic. Note that tensor product makes $RSp(\pi)$ into an $RO(\pi)$ module. Let $RU_0(\pi)$, $RO_0(\pi)$, and $RSp_0(\pi)$ be the augmentation ideals; these are the subgroups which consist of representations of virtual dimension zero. Let $g_n := e^{2\pi\sqrt{-1}/n}$ be the canonical generator of

$$\mathbb{Z}_n := \{\lambda \in \mathbb{C} : \lambda^n = 1\}$$

be the cyclic group of order n. Let $\rho_s(\lambda) = \lambda^s$ for s in the Poincare dual $\mathbb{Z}_n^* = \mathbb{Z}/n\mathbb{Z}$. The ρ_s are one dimensional representations of $\mathbb{Z}_n$ and $Irr(\mathbb{Z}_n) = \{\rho_s\}$. We have that

$$RU(\mathbb{Z}_n) = \oplus_{0 \leq s < n} \rho_s \cdot \mathbb{Z}.$$

The ring structure is given by the identity $\rho_s \otimes \rho_t = \rho_{s+t}$. Let $x := \rho_1 - \rho_0$. The identity

$$\rho_0 - \rho_s = (\rho_1 - \rho_0)(\rho_{s-1} + \rho_{s-2} + \ldots + \rho_0)$$

leads us the the algebraic structure

$$RU(\mathbb{Z}_n) = \mathbb{Z}[x]/\{(x+1)^n = 1\}, \text{ and}$$

$$RU_0(\mathbb{Z}_n) = x \cdot RU(\mathbb{Z}_n).$$

This will play a crucial role in what follows.

II.8 Locally Flat Bundles

Let $\tilde{M}$ be the universal cover of a connected manifold M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ by deck transformations. The bundle V_ρ associated with a representation $\rho : \pi_1(M) \to U(k)$ is defined by

$$V_\rho := \tilde{M} \times \mathbb{C}^k / \pi_1(M)$$

where we identify (x, v) with $(gx, \rho(g)v)$ for $g \in \pi_1(M)$. Note that the transition functions of such a bundle V_ρ are locally constant so V_ρ is flat. The bundle V_ρ has a natural inner product and a Riemannian connection ∇_ρ with zero curvature. The holonomy of the connection ∇_ρ is the representation ρ. Thus we may identify flat Hermitian vector bundles with unitary representations of the the fundamental group.

There is another way to look at this which is more topological in nature. Let $B\pi$ be the the classifying space of a finite group π. A π structure on a manifold M is a map f from M to $B\pi$ or equivalently a representation of $\pi_1(M)$ to π. If $\pi_1(M) = \pi$, there is a canonical π structure on M. If ρ is a unitary representation of π, then ρ defines a flat vector bundle $V_\rho(B\pi)$ over $B\pi$ and the pull back bundle $V_\rho(M) := f^*(V_\rho(B\pi))$ is a flat vector bundle over M which has a natural Riemannian connection with zero curvature.

II.9 Dirac Operators with Coefficients in Flat Bundles

Let P be an operator of Dirac type acting on $C^\infty(V)$. Let V_ρ be a flat bundle over M. Since the transition functions of V_ρ are locally constant, we can define the operator P_ρ on $C^\infty(V \otimes V_\rho)$ which is locally isomorphic to $\dim(\rho)$ copies of P. Let

$$\eta(P, \rho) := \eta(P_\rho) \in \mathbb{R}/\mathbb{Z}.$$

The map $\rho \mapsto \eta(P, \rho)$ is additive in ρ and extends to the representation ring $RU(\pi_1(M))$.

Let $P(\varepsilon)$ be a smooth 1 parameter family of operators of Dirac type and let $\rho \in RU(\pi_1(M))$. We apply Theorem 1.3. If m is even, then the variation $\dot{\eta}(P(\varepsilon), \rho)$ vanishes. Similarly, if m is odd and if $\rho \in RU_0(\pi_1(M))$, then the variation $\dot{\eta}(P(\varepsilon), \rho)$ also vanishes. Thus the eta invariant is a homotopy invariant in these situations.

As an example, take the operator $P = \sqrt{-1}\partial_\theta$ on the circle discussed in example I.8. Let g be the generator of $\pi_1(S^1) = \mathbb{Z}$. Since the fundamental group is Abelian, all the irreducible representations are 1 dimensional and are determined by their value on g. Let $\rho_\epsilon(g) = e^{2\pi\sqrt{-1}\epsilon}$. The corresponding operator is

$$P_\epsilon := e^{-\sqrt{-1}\epsilon\theta} P e^{\sqrt{-1}\epsilon\theta} = P - \epsilon.$$

Since $\eta(P_0) = 1/2$, we see that

$$\eta(P_\epsilon) = 1/2 + \epsilon \text{ in } \mathbb{R}/\mathbb{Z}, \text{ and } \eta(P, \rho_\epsilon - \rho_0) = \epsilon \text{ in } \mathbb{R}/\mathbb{Z}. \tag{2.2}$$

The calculation $\eta(P_\epsilon) = \epsilon + 1/2$ is **not** valid in $\mathbb{R}$; the eta invariant has integer jumps at the point $\epsilon \in \mathbb{Z}$. Let $P(u) := P - u$. The invariant $\eta(P(u), \rho_\epsilon - \rho_0)$ is independent of the parameter u in $\mathbb{R}/\mathbb{Z}$ but not in $\mathbb{R}$.

II.10 Twisted Spin Bordism Groups

Let ξ be a real vector bundle over the classifying space $B\pi$ of a finite group π. We consider a triple (M, s, f) where M is a smooth closed manifold of dimension m which need not be connected, where $f : M \to B\pi$ gives M a π structure, and where s is a *spin* structure on $T(M) \oplus f^*\xi$. The twisted bordism groups $MSpin_m(B\pi, \xi)$ is defined by imposing the equivalence relation $(M, s, f) \approx 0$ if there exists a compact smooth manifold N so that the boundary of N is M and so that the structures s and f extend over N. Notice that only the values of the first two Stiefel-Whitney classes $w_1(\xi)$ and $w_2(\xi)$ are important in the definition of $MSpin_m(B\pi, \xi)$.

Note that not every pair of cohomology classes (u_1, u_2) for $u_i \in H^i(B\pi; \mathbb{Z}_2)$ can be realized as the first two Stiefel Whitney classes of a vector bundle ξ. Nevertheless, there is a generalization of the twisted spin bordism groups defined above which associates an Abelian group to every such pair (u_1, u_2) which is isomorphic to $MSpin_m(B\pi, \xi)$ if $(u_1, u_2) = (w_1(\xi), w_2(\xi))$; see Stolz [39] for further details.

II.11 Geometric Twisted Spin Bordism Groups

Let $\tau = \tau(g) := R_{ijji}$ be the scalar curvature of a Riemannian metric g; we will discuss the scalar curvature further in §II.14 and in chapter 7. We now consider quadruples (M, s, f, g) where (M, s, f) are as above and where g is a metric of positive scalar curvature on M; necessarily $m \geq 2$. The geometric twisted spin bordism groups ${}^{+}MSpin_m(B\pi, \xi)$ are defined by introducing the equivalence relation $(M, s, f, g) \approx 0$ if there exists N as above such that the metric g extends as a metric of positive scalar curvature on N which is product near the boundary. The forgetful functor defines a natural homomorphism

$$ {}^{+}MSpin_m(B\pi, \xi) \to MSpin_m(B\pi, \xi). $$

This will play a crucial role in what follows.

II.12 Twisted Bordism Groups for BZ_ℓ

Henceforth let $\ell = 2^\nu > 1$ be a non-trivial power of 2. Let x and y be the non-zero elements of $H^1(B\mathbb{Z}_\ell; \mathbb{Z}_2) = \mathbb{Z}_2$ and $H^2(B\mathbb{Z}_\ell; \mathbb{Z}_2) = \mathbb{Z}_2$. We define:

a) Let ξ_0 be the trivial line bundle; $w_1(\xi_0) = 0$ and $w_2(\xi_0) = 0$.

b) Let ξ_1 be the real 2 bundle defined by the complex representation ρ_1; $w_1(\xi_1) = 0$ and $w_2(\xi_1) \neq 0$.

c) Let ξ_2 be the real line bundle defined by $\rho_{\ell/2}$; $w_1(\xi_2) \neq 0$ and $w_2(\xi_2) = 0$.

d) Let $\xi_3 = \xi_1 \oplus \xi_2$; $w_1(\xi_3) \neq 0$ and $w_2(\xi_3) \neq 0$.

Let M be a manifold whose universal cover admits a *spin* structure with fundamental group $\mathbb{Z}_\ell$. There exists $0 \leq i \leq 3$ and a suitable structure s so that $[(M, s, f)] \in MSpin_m(B\mathbb{Z}_\ell, \xi_i)$. If $\ell = 2$, $x^2 = y$; if $\ell > 2$, $x^2 = 0$.

a) We take $\xi = \xi_0$ if $w_1(M) = 0$ and $w_2(M) = 0$; M admits a *spin* structure.

b) We take $\xi = \xi_1$ if $w_1(M) = 0$ and $w_2(M) \neq 0$; M admits a *spin*^{c} structure with determinant line bundle given by ρ_1.

c) If $\ell = 2$, we take $\xi = \xi_3$ if $w_1(M) \neq 0$ and $w_2(M) = 0$; M admits a *pin*^{c} structure with determinant line bundle given by ρ_1.

d) If $\ell = 2$, we take $\xi = \xi_2$ if $w_1(M) \neq 0$ and $w_2(M) \neq 0$; M admits a *pin*$^+$ structure.

e) If $\ell > 2$, we take $\xi = \xi_2$ if $w_1(M) \neq 0$ and $w_2(M) = 0$; M admits a *pin*$^-$ structure.

f) If $\ell > 2$, we take $\xi = \xi_3$ if $w_1(M) \neq 0$ and $w_2(M) \neq 0$; M admits a *pin*^{c} structure with determinant line bundle given by ρ_1.

This will play a crucial role in what follows.

II.13 Connective K Theory

Let $\mathbb{HP}^2$ be the quaternionic projective plane with the usual homogeneous metric; this metric has positive scalar curvature. Let $p : E \to B$ be a geometrical fiber bundle with fiber $\mathbb{HP}^2$; we assume the transition functions lie in the group $PSp(3)$ of isometries of $\mathbb{HP}^2$. Since $\mathbb{HP}^2$ is simply connected, p_* is an isomorphism from $\pi_1(E)$ to $\pi_1(B)$. Thus any π structure on E is induced from a π structure on B. Let $T_m(B\mathbb{Z}_\ell, \xi) \subset MSpin_m(B\mathbb{Z}_\ell, \xi)$ be the subgroup generated by bordism classes (E, s, f_E) where f_E is the π structure on E induced from that on B. Let $Thom(\xi)$ be the Thom space of the k dimensional vector bundle ξ over $B\mathbb{Z}_\ell$. Define the twisted connective K theory groups by:

$$ko_m(B\mathbb{Z}_\ell, \xi) := \widetilde{ko}_{m+k}(Thom(\xi)).$$

The following result is due to Stolz [37] and is fundamental to our study.

2.3 Theorem. *Let π be a finite group.*

$$ko_m(B\pi,\xi)_{(2)} \approx \{MSpin_m(B\pi,\xi)/T_m(B\pi,\xi)\}_{(2)}.$$

If $\pi = \mathbb{Z}_\ell$, the groups in question are 2 primary and therefore we do *not* have to localize at the prime 2. Thus Theorem 2.3 implies:

$$ko_m(B\mathbb{Z}_\ell,\xi) \approx MSpin_m(B\mathbb{Z}_\ell,\xi)/T_m(B\mathbb{Z}_\ell,\xi).$$

We define ko_m^+ to be the image of ${}^+MSpin_m$:

$$ko_m^+(B\mathbb{Z}_\ell,\xi) := {}^+MSpin_m(B\mathbb{Z}_\ell,\xi)/T_m(B\mathbb{Z}_\ell,\xi).$$

This is generated by manifolds which admit metrics of positive scalar curvature.

II.14 The Lichnerowicz Formula

Let D be the Dirac operator on a compact *spin* manifold M. Lichnerowicz [28] generalized the Weitzenböch formulas to show

$$D^2 = -\operatorname{Tr}(\nabla^2) + \tau/4$$

where ∇ is the spinor Laplacian and τ is the scalar curvature. One can use this formula to compute

$$|D\phi|_{L^2}^2 = |\nabla\phi|_{L^2}^2 + \int_M (\tau\phi,\phi)/4dvol.$$

Therefore if the metric in question has positive scalar curvature, then there are no harmonic spinors. This observation extends without change to the case that M admits a $spin^c$ structure with flat determinant line bundle as the curvature of the determinant line bundle is zero. If M has a smooth boundary and if we impose spectral boundary conditions (i.e. boundary conditions of Atiyah-Patodi-Singer type), the analysis is a bit more complicated but again there are no harmonic spinors. See Botvinnik and Gilkey [8, 10] for details.

II.15 Extending the Eta Invariant to Connective K Theory

Let π be a finite group and let $[(M,s,f)] \in MSpin_m(B\pi,\xi)$. Suppose first m is odd. We shall assume that ξ admits a *spin*^{c} structure with flat associated line bundle. Let P be the Dirac operator defined by this structure, see Gilkey [20] for details. We let $\rho \in RU_0(\pi)$ define the eta invariant $\eta(M,s,f,g)(\rho)$ where we choose a Riemannian metric g on M. Suppose next that m is even. We assume that ξ admits a *pin*^{c} structure with flat associated line bundle and that ξ carries the orientation of M i.e. $w_1(f^*\xi) = w_1(M)$. Let P be the Dirac operator defined by this structure, again see Gilkey [20] for details. We let $\rho \in RU(\pi)$ define the eta invariant $\eta(M,s,f,g)(\rho)$.

2.4 Theorem.

(1) *Let m be odd and let $i = 0,1$. If $m = 4k+1$, let $\rho \in RU(\mathbb{Z}_\ell)$, and if $m = 4k-1$, let $\rho \in RU_0(\mathbb{Z}_\ell)$. We can extend the map $M \mapsto \eta(M,s,f,g)(\rho)$ to a homomorphism η_ρ from $ko_m(B\mathbb{Z}_\ell,\xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 3$ mod 8, if $i = 0$, and if ρ is real, we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.*

(2) *Let m be even and let $i = 2,3$. Let $\rho \in RU(\mathbb{Z}_\ell)$. We can extend the map $M \mapsto \eta(M,s,f,g)(\rho)$ to a homomorphism η_ρ from $ko_m(B\mathbb{Z}_\ell,\xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 2$ mod 8, if $i = 2$, and if ρ is real, we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.*

Proof. We use Theorem 1.4. Suppose M is the boundary of a compact manifold and the structures extend over N. If m is odd, let D_ρ be *spin*^{c} complex over N with coefficients in ρ; if m is even, let D_ρ be the *pin*^{c} complex over N with coefficients in ρ. Then modulo a possible sign convention which plays no role, the tangential operator of D_ρ is the Dirac operator P_ρ on M with coefficients in ρ.

Note that the local invariants satisfy the relationship $\mathcal{P}(D_\rho) = \dim(\rho)\mathcal{P}(D)$. We have $\mathcal{P}(D_\rho)$ vanishes for dimensional reasons if m is even. Also $\mathcal{P}(D_\rho)$ vanishes if we have $m = 4k+1$ since the *spin*^{c} structure is flat. If $m = 4k-1$, then $\dim(\rho)\mathcal{P}(D_\rho)$ vanishes since we assumed $\dim(\rho) = 0$. Thus the index theorem shows $\eta(P_\rho, s)/2|_{s=0} \in \mathbb{Z}$ so $\eta(P, s, f, g)(\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ and the eta invariant extends to the equivariant spin bordism groups in question. In certain cases, the operators admit quaternion structures so $\operatorname{index}(D_\rho) \in 2\mathbb{Z}$ and this $\eta(P_\rho, s)/2|_{s=0} \in 2\mathbb{Z}$. In these cases, we can extend $\eta(M, \rho)$ to take values in $\mathbb{R}/2\mathbb{Z}$.

Let E be the total space of a $\mathbb{HP}^2$ fibration. Botvinnik and Gilkey [11] showed that E admits a metric g so that $\eta(E, s, f, g)(\rho)$ vanishes in $\mathbb{R}$ and so that g has positive scalar curvature. This shows the eta invariant vanishes on the subgroups $T_m(B\pi, \xi_i)$ and by Theorem 2.3 extends to connective K theory. We refer to [20] for further details. $\square$

II.16 Extending the Eta Invariant to Geometric Bordism

We refer to Botvinnik and Gilkey [8, 10] for the proof of the following result; in the papers cited, the authors dealt with the orientable case. However, the extension to the non orientable setting is immediate and is therefore omitted.

2.5 Theorem. *Let π be a finite group. Let ρ be a virtual representation of π and let ξ be a real vector bundle over the classifying space $B\pi$. If m is even, assume that ξ is non-orientable; if m is odd, assume that ξ is orientable. Assume that ξ admits a (s)pin^c structure with flat determinant line. Then we can extend the map $M \mapsto \eta(M, s, f, g)(\rho)$ to a homomorphism η_ρ from ${}^{+}MSpin_m(B\mathbb{Z}_\ell, \xi)$ to $\mathbb{R}$.*

II.17 The $\hat{A}$ Genus

Let g be a Riemannian metric on a closed manifold M of dimension m. Let $P(s,g)$ be the Dirac operator defined by a *spin* structure s. If $m \equiv 0 \bmod 4$, we can split $P(s,g) = P^+(s,g) + P^-(s,g)$ into the chiral operators of the spin complex. If $m \equiv 0 \bmod 4$, let

$$\hat{A}(M,s,g) := \dim\ker(P^+(s,g)) - \dim\ker(P^-(s,g)) \in \mathbb{Z}$$

be the index of the spin complex; this takes values in $\mathbb{Z}$ if $m \equiv 0 \bmod 8$ and values in $2\mathbb{Z}$ if $m \equiv 4 \bmod 8$. The index theorem of Atiyah and Singer [5] shows there exists a polynomial $\mathcal{A}$ in the Pontrjagin forms on M so that $\hat{A}(M) = \int_M \mathcal{A}$. Consequently $\hat{A}(M) = \hat{A}(M,s,g)$ is independent of the metric g and the *spin* structure s. If $m \equiv 2 \bmod 4$, then the index of the spin complex vanishes. If $m \equiv 2 \bmod 8$, we define

$$\hat{A}(M,s,g) := \dim\ker(P^+(s,g)) = 1/2 \dim\ker(P(s,g)) \in \mathbb{Z}_2.$$

One can show that $\hat{A}(M,s) = \hat{A}(M,s,g)$ is independent of the metric g and only depends on the *spin* structure s. Finally, if $m \equiv 1 \bmod 8$, we define

$$\hat{A}(M,s,g) = \dim(\ker(P(s,g)) \in \mathbb{Z}_2;$$

again $\hat{A}(M,s) = \hat{A}(M,s,g)$ is independent of g. We set $\hat{A} = 0$ for other values of m.

Let $[(M,s,f)] \in MSpin(B\mathbb{Z}_\ell, \xi_i)$. Suppose that $i = 0$. If $m \equiv 1, 2 \bmod 8$, let

$$\hat{\mathbb{A}}(M,s,f) = \hat{A}(M,s) \oplus \hat{A}(M,s_L) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where s_L is the spin structure defined by twisting the given spin structure s by the orientation line bundle defined by the representation $\rho_{\ell/2}$; otherwise let $\hat{\mathbb{A}}(M,s,f) = \hat{A}(M)$. If $i \neq 0$, let $\hat{\mathbb{A}}(M,s,f) = \hat{A}(\tilde{M})$ where $\tilde{M}$ is the associated principal $\mathbb{Z}_\ell$ bundle. We can extend the $\hat{\mathbb{A}}$ genus to the equivariant bordism groups. Since $\hat{\mathbb{A}}$ vanishes on T_m, we see that $\hat{\mathbb{A}}$ also extends to connective K theory.

II.18 The Circle

Let s_0 be the spin structure on the circle S^1 obtained by regarding S^1 as the boundary of the disk D^2. The associated principal spin bundle is the connected 2 fold covering of the principal SO bundle; we have $P_{SO}(S^1) = S^1$. The real spinor bundle L is the Möbius line bundle. Sections to L take the form $f(\theta)$ where $f(2\pi) = -f(0)$. The Dirac operator defined by the spin structure s_0 is given by $P(s_0) = -\sqrt{-1}\partial_\theta - 1/2$ relative to a suitable trivialization of the complexification of L. Thus a spectral resolution of $P(s_0)$ is given by $\{e^{\sqrt{-1}(n+1/2)\theta}\}$ for $n \in \mathbb{Z}$. Thus $\dim\ker(P(s_0)) = 0$ and $\hat{A}(S^1, s_0) = 0$. Since the $\hat{A}$ genus is a bordism invariant and (S^1, s_0) bounds, we must have $\hat{A}(S^1, s_0) = 0$. We have that $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$. Let s_1 be the other spin structure; this is s_0 with coefficients in L. Thus the associated real spinor bundle is $L \otimes L = 1$. The associated principal spin bundle is the disconnected 2 fold covering of the principal SO bundle. We have $P(s_1) = -\sqrt{-1}\partial_\theta$ and a spectral resolution of $P(s_1)$ is given by $\{e^{\sqrt{-1}n\theta}\}$ for $n \in \mathbb{Z}$. Thus $\dim\ker(P(s_1)) = 1$ and $\hat{A}(S^1, s_1) = 1$. Since the $\hat{A}$ genus is a bordism invariant and $\hat{A}(S^1, s_1) \neq 0$, (S^1, s_1) does not bound.

There is a point of possible epistemological confusion here. We choose coordinate systems $\mathcal{O}_1 := (0, 2\pi)$ and $\mathcal{O}_2 := (\pi, 3\pi)$ for S^1. If we choose the canonical trivialization of S^1, the transition function $\phi_{12} = 1$. If we choose a lift $\tilde{\phi}_{12} = 1$ to $Spin(2) = S^1$, this gives S^1 the *trivial* spin structure; this is the *non-bounding* spin structure s_1. If we choose a lift $\tilde{\phi}_{12} = 1$ on $(0, \pi) = (2\pi, 3\pi)$ and $\tilde{\phi}_{12} = -1$ on $(\pi, 2\pi)$, we define the Möbius spin structure; this is the *bounding* spin structure.

Let $f_0 : \pi_1(S^1) \to \mathbb{Z}_\ell$ be the trivial map; this gives S^1 the trivial $\mathbb{Z}_\ell$ structure. Let $f_1 : \pi_1(S^1) \to \mathbb{Z}_\ell$ be reduction mod ℓ; this gives S^1 the canonical $\mathbb{Z}_\ell$ structure.

Let ℓ be even. Let $\hat{\mathbb{A}}$ be the equivariant $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ valued A-roof genus. It is immediate from the discussion that we have given above that the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ genus $\hat{\mathbb{A}}$ is given by

$$\begin{aligned} &\hat{\mathbb{A}}(S^1, s_0, f_0) = 0 \oplus 0,\ \hat{\mathbb{A}}(S^1, s_1, f_0) = 1 \oplus 1, \\ &\hat{\mathbb{A}}(S^1, s_0, f_1) = 0 \oplus 1,\ \hat{\mathbb{A}}(S^1, s_1, f_1) = 1 \oplus 0. \end{aligned} \tag{2.6}$$

We suppress the metric from the notation. Let $\rho_0(g) = 1$ and $\rho_s(g) = e^{2\pi\sqrt{-1}s/\ell}$. We use equation (2.2) to compute the $\mathbb{R}/\mathbb{Z}$ valued eta invariant defined in §II.15

$$\begin{aligned} &\eta(S^1, s_0, f_0)(\rho_s) = 0,\ \eta(S^1, s_1, f_0)(\rho_s) = 1/2, \\ &\eta(S^1, s_0, f_1)(\rho_s) = s/\ell,\ \eta(S^1, s_1, f_1)(\rho_s) = s/\ell + 1/2. \end{aligned} \tag{2.7}$$

2.8 Lemma.

(1) $ko_1(B\mathbb{Z}_\ell) = MSpin_1(B\mathbb{Z}_\ell) = \mathbb{Z}_2 \oplus \mathbb{Z}_\ell$.

(2) $\widetilde{ko}_1(B\mathbb{Z}_\ell) = \widetilde{MSpin}_1(B\mathbb{Z}_\ell) = \mathbb{Z}_\ell$.

(3) $ko_1(B\mathbb{Z}_\ell) \cap \ker(\hat{\mathbb{A}}) = MSpin_1(B\mathbb{Z}_\ell) \cap \ker(\hat{\mathbb{A}}) = \mathbb{Z}_{\ell/2}$.

(4) $ko_1(B\mathbb{Z}_\ell, \xi_1) = MSpin_1(B\mathbb{Z}_\ell, \xi_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_\ell$.

(5) *If* $M^1 \in ko_1(B\mathbb{Z}_\ell)$, *then* $\ell\eta(M^1, s, f)(\rho_0 - \rho_1) = \hat{A}(M^1, s_L)$ *mod* $2\mathbb{Z}$.

Proof. Since $T_m = 0$ for $m < 8$, we may identify $MSpin_m$ with ko_m in these dimensions. It follows from Table 6.1 that $|MSpin_1(\mathbb{Z}_\ell)| = 2\ell$ and $|MSpin_1(\mathbb{Z}_\ell, \xi_1)| = 2\ell$; this uses the Adams spectral sequence. The invariants in equation (2.7) are bordism invariants by Theorem II.15 and the manifolds in question determine elements of $ko_1(B\mathbb{Z}_\ell, \xi_i)$ for $i = 0, 1$. The first and third assertions now follow; the remaining assertions follow from equations (II.18) and (2.7) and from the observation $ko_1(pt) = MSpin_1(pt) = \mathbb{Z}_2$. $\square$

II.19 The Torus

The torus $T^2 = S^1 \times S^1$ has 4 inequivalent spin structures. Let s_1 be the product spin structure induced from the trivial (i.e. non-bounding) spin structure on the circle. The associated spinor bundle is a trivial 2 plane bundle and the Dirac Laplacian takes the form $D = -\partial^2_{\theta_1} - \partial^2_{\theta_2}$. Thus $\dim\ker(D) = 2$ and $\hat{A}(T^2, s_1) = 1$. Let $f_1 : \pi_1(T^2) \to \mathbb{Z}_\ell$ be any surjective map; this gives T^2 a non-trivial $\mathbb{Z}_\ell$ structure. Then the Dirac Laplacian is twisted by a suitable Möbius line bundle and has no kernel and $\hat{A}(T^2, s_L) = 0$. Let f_0 be the trivial $\mathbb{Z}_\ell$ structure and let $\hat{\mathbb{A}}$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ valued genus. We have

$$\begin{aligned} &\hat{\mathbb{A}}(T^2, s_1, f_0) = 1 \oplus 1,\ \hat{\mathbb{A}}(T^2, s_1, f_1) = 1 \oplus 0, \\ &\hat{\mathbb{A}}(T^2, s_L, f_0) = 0 \oplus 0\ ,\hat{\mathbb{A}}(T^2, s_L, f_1) = 0 \oplus 1. \end{aligned} \tag{2.9}$$

Again, results we shall discuss in §3 show that $|MSpin_2(B\mathbb{Z}_\ell)| \leq 4$ which shows:

2.10 Lemma. *We have $ko_2(B\mathbb{Z}_\ell) = MSpin_2(B\mathbb{Z}_\ell) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

II.20 An Obstruction

If M admits a metric of positive scalar curvature, there are no harmonic spinors, see §II.14. Thus $\hat{A}(M) = 0$. Therefore if $\hat{A}(M) \neq 0$, M does not admit a metric of positive scalar curvature. The Kummer surface

$$K^4 := \{\langle z\rangle \in \mathbb{C}P^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

is an algebraic surface which admits a *spin* structure with $\hat{A}(K^4) = 2$. Thus K^4 does not admit a metric of positive scalar curvature.

II.21 Cartesian Product

Let $MSpin_m$ be the ordinary *spin* bordism groups. Cartesian product makes $ko_*(B\pi, \xi)$ and ${}^+MSpin_*(B\pi, \xi)$ into $MSpin_*$ modules. Let $M \in ko_m(B\pi, \xi)$ or $M \in {}^+MSpin_m(B\pi, \xi)$ and let $N \in MSpin_{8k}$. One then has

$$\begin{aligned} \eta_\rho(M \times N) &= \eta_\rho(M) \times \hat{A}(N) \\ \hat{A}(M \times N) &= \hat{A}(M) \times \hat{A}(N). \end{aligned} \tag{2.11}$$

CHAPTER III

LENS SPACES AND LENS SPACE BUNDLES

III.1 Introduction

The lens spaces and lens space bundles form the geometric generators for the groups $ko_m(B\mathbb{Z}_\ell, \xi_i) \cap \ker(\hat{A})$ if m is odd for $i = 0, 1$. In this section, we define these manifolds and discuss their basic properties

3.1 Definition. Let $\tau(\vec{a}) := \rho_{a_1} \oplus ... \oplus \rho_{a_k}$ be representation of $\mathbb{Z}_\ell$ in $U(k)$. If $\vec{a} = (a_1, ..., a_k)$ is a collection of odd integers, this defines a fixed point free action of $\mathbb{Z}_\ell$ on the unit sphere S^{2k-1} in $\mathbb{R}^{2k} = \mathbb{C}^k$. The lens space is the quotient manifold

$$L^{2k-1}(\ell; \vec{a}) := S^{2k-1}/\tau(\vec{a})(\mathbb{Z}_\ell).$$

Let $H^{\otimes 2} \oplus (k-1)1$ be the Whitney sum of the tensor square of the complex Hopf line bundle with $(k-1)$ copies of the trivial complex line bundle over complex projective space $\mathbb{CP}^1$ which we identify with the sphere S^2. Let $\lambda \in S^1$ act by multiplication by λ^{a_ν} on the ν^{th} summand. This action restricts to a fixed point free action of $\mathbb{Z}_\ell$ on the sphere bundle $S(H^{\otimes 2} \oplus (k-1)1)$. Let

$$X^{2k+1}(\ell; \vec{a}) := S(H^{\otimes 2} \oplus (k-1)1)/\tau(\vec{a})(\mathbb{Z}_\ell).$$

This manifold is a lens space bundle over S^2. These manifolds admit metrics of positive scalar curvature if the dimension is at least 3; see, for example, [11].

Recall the notation of §II.12. Then

(1) $\mathbb{RP}^{4k} \in MSpin_{4k}(B\mathbb{Z}_2, \xi_3)$.

(2) $\mathbb{RP}^{4k+2} \in MSpin_{4k+2}(B\mathbb{Z}_2, \xi_2)$

(3) $L^{4k+1}(\ell; -) \in MSpin_{4k+1}(B\mathbb{Z}_\ell, \xi_1)$.

(4) $L^{4k+3}(\ell; -) \in MSpin_{4k+3}(B\mathbb{Z}_\ell, \xi_0)$.

(5) $X^{4k+1}(\ell; -) \in MSpin_{4k+1}(B\mathbb{Z}_\ell, \xi_0)$.

(6) $X^{4k+3}(\ell; -) \in MSpin_{4k+3}(B\mathbb{Z}_\ell, \xi_1)$.

There are combinatorial formulas for the eta invariant.

(1) If k is even, let $\mathcal{F}_L(\vec{a}; \lambda) = \lambda^{-|\vec{a}|/2} \det(I - \tau(\vec{a})(\lambda))$.

(2) If k is odd, let $\mathcal{F}_L(\vec{a}; \lambda) = \lambda^{-(|\vec{a}|+1)/2} \det(I - \tau(\vec{a})(\lambda))$.

(3) If $\lambda \neq 1$, let $\mathcal{G}_L(\vec{a}; \lambda) = \mathcal{F}_L(\vec{a}; \lambda)^{-1}$. If $\lambda = 1$, let $\mathcal{G}_L(\vec{a}; \lambda) = 0$.

(4) Let $\mathcal{G}_X(\vec{a}; \lambda) = (1 + \lambda^{a_1})(1 - \lambda^{a_1})^{-1} \mathcal{G}_L(\vec{a}; \lambda)$

We refer to [11, 13] for the proof of the following result; the assertions concerning the eta invariant are based on results of Donnelly [15].

3.2 Lemma. *Let $\Sigma_\lambda := \Sigma_{\lambda \in \mathbb{Z}_\ell}$, and let $\tilde{\Sigma}_\lambda := \Sigma_{\lambda \in \mathbb{Z}_\ell, \lambda \neq 1}$.*

(1) *For $m \geq 3$, $L^m(\ell; \vec{a})$ and $X^m(\ell; \vec{a})$ admit metrics of positive scalar curvature.*

(2) *If k is even, then $L^{2k-1}(\ell; \vec{a})$ and $X^{2k+1}(\ell; \vec{a})$ admit spin structures.*

(3) *If k is odd, then $L^{2k-1}(\ell; \vec{a})$ and $X^{2k+1}(\ell; \vec{a})$ have spin^c structures with determinant line bundle given by ρ_1.*

(4) *We have $\eta(L^{2k-1}(\ell; \vec{a}))(\rho) = \ell^{-1} \tilde{\Sigma}_\lambda Tr(\rho) \mathcal{G}_L(\vec{a}; \lambda) \in \mathbb{Q}$.*

(5) *We have $\eta(X^{2k+1}(\ell; \vec{a}))(\rho) = \ell^{-1} \tilde{\Sigma}_\lambda Tr(\rho) \mathcal{G}_X(\vec{a}; \lambda) \in \mathbb{Q}$.*

We have the following integrality theorem; we refer to [11] for the proof.

3.3 Lemma. *Let $\rho \in RU_0(\mathbb{Z}_\ell)^j$. Let $m < 2j + 1$. Then*

$$\eta(L^m(\ell; \cdot))(\rho) \in \mathbb{Z} \text{ and } \eta(X^m(\ell; \cdot))(\rho) \in \mathbb{Z}.$$

III.2 The Eta Invariant

The Poincare dual A^* of an Abelian group A is the group of homomorphisms from A to $\mathbb{R}/\mathbb{Z}$. Thus, for example, $\mathbb{Z}/\ell\mathbb{Z}$ is the Poincare dual of $\mathbb{Z}_\ell$. Let $\eta^*(M)$ be the homomorphism which sends ρ to $\eta(M)(\rho)$. By Theorem 2.4, the eta invariant extends to connective K theory:

$$\eta^* : ko_{4k+1}(B\mathbb{Z}_\ell, \xi_i) \to RU(\mathbb{Z}_\ell)^*, \text{ and } \eta^* : ko_{4k+3}(B\mathbb{Z}_\ell, \xi_i) \to RU_0(\mathbb{Z}_\ell)^*.$$

Let $\theta(M) = \eta(M)(\rho_0 - \rho_{\ell/2}) \in \mathbb{R}/2\mathbb{Z}$ if $\xi = \xi_0$ and if $4k+3 \equiv 3 \bmod 8$. Set $\theta = 0$ otherwise.

III.3 Generators for $ko_{4k+1}(BZ_\ell, \xi_i)$ for $i = 0, 1$.

Botvinnik and Gilkey [11] established some technical results concerning the eta invariant in order to discuss the Gromov-Lawson conjecture for manifolds with cyclic fundamental groups with spin universal cover. We shall apply their results in this section to study connective K theory; we will also apply these results later in §7 to discuss the Gromov-Lawson conjecture in the non-orientable setting.

In the free Abelian group generated by lens spaces and lens space bundles which have at least one index "3", define:

(1) $\mathcal{B} : L^m(\ell; \cdot, 3) \mapsto L^m(\ell; \cdot, 1) - 3L^m(\ell; \cdot, 3)$.

(2) $M^L_{m,j} := \mathcal{B}^j L(\ell; 3, ..., 3)$.

(3) $\mathcal{B} : X^m(\ell; \cdot, 3) \mapsto X^m(\ell; \cdot, 1) - 3X^m(\ell; \cdot, 3)$.

(4) $M^X_{m,j} := \mathcal{B}^j X(\ell; 1, 3, 3, ..., 3)$.

We define $M^L_{m,j}$ for $2j - 1 \leq m$. When considering the lens space bundles, we assume the index "3" in question is not the first index. Thus we define $M^X_{m,j}$ for $2j-1 \leq m-4$. The eta invariant is additive with respect to direct sums and extends to this setting.

3.4 Lemma. *Let $\rho \in RU(\mathbb{Z}_\ell)$ and let $\psi := (\rho_0 - \rho_1)^2\rho_{-1}$.*

(1) *$\eta(\mathcal{B}M)(\rho) = \eta(M)(\psi\rho)$ for M a lens space or suitable lens space bundle.*

(2) *$\eta(M^L_{m,j})(\rho) = \eta(M^L_{m,0})(\psi^j\rho)$ and $\eta(M^X_{m,j})(\rho) = \eta(M^X_{m,0})(\psi^j\rho)$.*

Proof. We use Lemma 3.2 to see that

$$\mathcal{G}_L(\vec{a}, 1; \lambda) - 3\mathcal{G}_L(\vec{a}, 3; \lambda) = \psi(\lambda)\mathcal{G}_L(\vec{a}, 3; \lambda).$$

Consequently

$$\eta(\mathcal{B}L^{2k+1}(\ell; \vec{a}, 3))(\rho) = \eta(L^{2k+1}(\ell; \vec{a}, 3)(\psi\rho)$$

and assertions concerning lens spaces follow. Similarly

$$\eta(\mathcal{B}X^{2k+3}(\ell; \vec{a}, 3))(\rho) = \eta(X^{2k+3}(\ell; \vec{a}, 3)(\psi\rho)$$

provided that the index "3" is not the first index; the first index plays a distinguished role in the definition of $\mathcal{G}_X$. $\square$

Let $\alpha := \rho_{-3}(\rho_0 - \rho_3)^2 \in RU_0(B\mathbb{Z}_\ell)^2$. Then $\operatorname{Tr}\alpha(\lambda) := \mathcal{F}_L(3, 3; \lambda)$. The homomorphism which sends ρ to $\alpha\rho$ defines a dual map α^* from $RU_0(\mathbb{Z}_\ell)^*$ to $RU(\mathbb{Z}_\ell)^*$ and from $RU_0(\mathbb{Z}_\ell)^*$ to $RU_0(\mathbb{Z}_\ell)^*$.

3.5 Lemma.

(1) $\eta(M^L_{m,j})(\alpha\rho) = \eta(M^L_{m-4,j})(\rho)$.

(2) $\eta(M^X_{m,j})(\alpha\rho) = \eta(M^X_{m-4,j})(\rho)$.

(3) $\eta(M^L_{5,0})(\alpha\rho_{-2}) = (\ell - 1)/2\ell$.

(4) $\eta(M^X_{5,0})(\alpha(\rho_0 - \rho_3)\rho_{-2}) = (\ell - 2)/\ell$.

(5) *If $\rho \in RU(\mathbb{Z}_\ell)$, then $\eta(M^L_{4k+1,k})(\alpha\rho) \in \mathbb{Z}$. Furthermore there exists γ^L_{4k+1} so that $\eta(M^L_{4k+1,k})(\alpha\rho)(\gamma^L_{4k+1}) = (\ell - 1)/2\ell$.*

(6) *If $\rho \in RU(\mathbb{Z}_\ell)$, then $\eta(M^X_{4k+1,k})(\alpha\rho) \in \mathbb{Z}$. Furthermore there exists γ^X_{4k+1} so that $\eta(M^L_{4k+1,k})(\alpha\rho)(\gamma^X_{4k+1}) = (\ell - 2)/\ell$.*

Proof. Since $\mathcal{F}(\vec{a},3,3;\lambda)=\alpha(\lambda)\mathcal{F}(\vec{a};\lambda)$,

$$\eta(L^{m+4}(\ell;\vec{a},3,3))(\alpha\rho)=\eta(L^m(\ell;\vec{a}))(\rho)$$

$$\eta(X^{m+4}(\ell;\vec{a},3,3))(\alpha\rho)=\eta(X^m(\ell;\vec{a}))(\rho).$$

The first two assertions now follow. We prove the second two assertions by computing:

$$\begin{aligned}\eta(M^L_{5,0})(\alpha\rho_{-2})&=\ell^{-1}\tilde{\Sigma}_\lambda(1-\lambda^3)^{-1}\\ &=(2\ell)^{-1}\tilde{\Sigma}_\lambda\{(1-\lambda)^{-1}+(1-\bar{\lambda})^{-1})\\ &=(2\ell)^{-1}\tilde{\Sigma}_\lambda 1=(\ell-1)/(2\ell),\\ \eta(M^X_{5,0})(\alpha(\rho_0-\rho_3)\rho_{-2})&=\ell^{-1}\tilde{\Sigma}_\lambda(1+\lambda^3)\\ &=\ell^{-1}\tilde{\Sigma}_\lambda(1+\lambda)=(\ell-2)/\ell.\end{aligned}$$

We complete the proof by establishing the final two assertions. We use Lemma 3.4 to compute

$$\eta(M^L_{4k+1,k})(\alpha\rho)=\eta(M^L_{4k+1,0})(\alpha\psi^k\rho),\ \text{and}$$

$$\eta(M^X_{4k+1,k})(\alpha\rho)=\eta(M^X_{4k+1,0})(\alpha\psi^k\rho).$$

Then $\rho\alpha\psi^k\in RU_0(\mathbb{Z}\ell)^{2k+2}$. Since $\dim(M^L_{4k+1,0})=\dim(M^X_{4k+1,0})=2(2k+1)-1$, these eta invariants take values in $\mathbb{Z}$ by Lemma 3.3. Similarly, we compute:

$$\eta(M^L_{4k+1,k})(\gamma_{k,L})=\eta(M^L_{m,0})(\gamma_{k,L}\psi^k),\ \text{and}$$

$$\eta(M^X_{4k+1,k})(\gamma_{k,X})=\eta(M^X_{m,0})(\gamma_{k,X}\psi^k).$$

We have $\psi^kR(\mathbb{Z}_\ell)=\alpha^kR(\mathbb{Z}_\ell)$. Thus we may choose $\gamma_{k,L}$ so that $\gamma_{k,L}\psi^k=\alpha^k\rho_{-2}$; let $\gamma_{k,X}=\gamma_{k,L}(\rho_0-\rho_3)$. Then

$$\begin{aligned}\eta(M^L_{m,0})(\gamma_{k,L}\psi^k)&=\eta(M^L_{m,0})(\alpha^k\rho_{-2})=\eta(M^L_{5,0})(\alpha\rho_{-2})\\ &=(\ell-1)/2\ell\\ \eta(M^X_{m,0})(\gamma_{k,X}\psi^k)&=\eta(M^L_{m,0})(\alpha^k(\rho_0-\rho_3)\rho_{-2})\\ &=\eta(M^L_{5,0})(\alpha(\rho_0-\rho_3)\rho_{-2})=(\ell-2)/\ell.\quad\square\end{aligned}$$

Let $k \geq 0$. We define

(1) $\mathcal{M}^L_{4k-1}(\ell) := \operatorname{span}_{0\leq j\leq 2k}\{M^L_{4k-1,j}\} \subset ko_{4k-1}(B\mathbb{Z}_\ell, \xi_0)$.

(2) $\mathcal{M}^L_{4k+1}(\ell) := \operatorname{span}_{0\leq j\leq 2k+1}\{M^L_{4k+1,j}\} \subset ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1)$.

(3) $\mathcal{M}^X_{4k-1}(\ell) := \operatorname{span}_{0\leq j\leq 2k-2}\{M^X_{4k-1,j}\} \subset ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1)$.

(4) $\mathcal{M}^X_{4k+1}(\ell) := \operatorname{span}_{0\leq j\leq 2k-1}\{M^X_{4k+1,j}\} \subset ko_{4k+1}(B\mathbb{Z}_\ell, \xi_0)$.

3.6 Lemma. *Let $k \geq 0$ and if $\ell \geq 4$.*

(1) $|\eta^*\mathcal{M}^L_{4k+5}(\ell)| \geq (2\ell)^{k+2}$.

(2) $|\eta^*\mathcal{M}^X_{4k+5}(\ell)| \geq (\ell/2)^{k+2}$.

(3) $ko_{4k+5}(B\mathbb{Z}_\ell, \xi_0) \cap \ker \hat{A} = \mathcal{M}^X_{4k+5}(\ell)$.

(4) $ko_{4k+5}(B\mathbb{Z}_\ell, \xi_1) = \mathcal{M}^L_{4k+5}(\ell)$.

Proof. It is immediate that

$$|\eta^*\mathcal{M}^L_m(\ell)| \geq |\alpha^*\eta^*\mathcal{M}^L_m(\ell)| \cdot |\ker \alpha^* \cap \eta^*\mathcal{M}^L_m(\ell)|$$

$$|\eta^*\mathcal{M}^X_m(\ell)| \geq |\alpha^*\eta^*\mathcal{M}^X_m(\ell)| \cdot |\ker \alpha^* \cap \eta^*\mathcal{M}^X_m(\ell)|.$$

We use Lemma 3.5 to see that

$$|\alpha^*\eta^*\mathcal{M}^L_m(\ell)| \geq |\eta^*\mathcal{M}^L_{m-4}(\ell)|,$$

$$|\alpha^*\eta^*\mathcal{M}^X_m(\ell)| \geq |\eta^*\mathcal{M}^X_{m-4}(\ell)|,$$

$$|\alpha^*\eta^*\mathcal{M}^L_5(\ell)| \geq 2\ell,\ \ |\alpha^*\eta^*\mathcal{M}^X_5(\ell)| \geq \ell/2,$$

$$|\ker \alpha^* \cap \eta^*\mathcal{M}^L_m(\ell)| \geq 2\ell,\ \ |\ker \alpha^* \cap \eta^*\mathcal{M}^X_m(\ell)| \geq \ell/2.$$

This proves the first two assertions and gives a lower bound for $ko_m(B\mathbb{Z}_\ell, \xi_i)$ for $i = 0, 1$ if $m \equiv 1 \bmod 4$.

We let $to_m(B\mathbb{Z}_\ell, \xi_i) := \ker(\hat{A}) \cap ko_m(B\mathbb{Z}_\ell, \xi_i)$ and $|\mathcal{T}(m, \xi_i)| := |to_m(B\mathbb{Z}_\ell, \xi_i)|$. The following estimates were established in Botvinnik and Gilkey [11].

Table 3.7

	$\mathcal{T}(m,\xi_0)$	$\mathcal{T}(m,\xi_1)$	$\mathcal{T}(m,\xi_2)$	$\mathcal{T}(m,\xi_3)$
$m=8k$	1	1	2^{2k}	2^{2k+1}
$m=8k+1$	$(\ell/2)^{2k+1}$	$(2\ell)^{2k+1}$	1	1
$m=8k+2$	1	1	2^{2k+2}	2^{2k+1}
$m=8k+3$	$2(2\ell)^{2k+1}$	$(\ell/2)^{2k+1}$	1	1
$m=8k+4$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+5$	$(\ell/2)^{2k+2}$	$(2\ell)^{2k+2}$	1	1
$m=8k+6$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+7$	$(2\ell)^{2k+2}$	$(\ell/2)^{2k+2}$	1	1

The final two assertions follow from these estimates and from the estimates of (1) and (2). □

III.4 Generators for $ko_{4k+3}(BZ_\ell,\xi_i)$ for $i=0,1$.

Again, we begin our discussion with a technical Lemma.

3.8 Lemma.

(1) $\eta(M^L_{3,0})((\rho_0-\rho_3)\rho_{-2})=(\ell-1)/2\ell.$

(2) $\eta(M^X_{3,0})((\rho_0-\rho_3)^2\rho_{-2})=(\ell-2)/\ell.$

Proof. We prove the first two assertions by computing:

$$\eta(M^L_{3,0})((\rho_0-\rho_3)\rho_{-2})=\ell^{-1}\tilde\Sigma_\lambda(1-\lambda^3)^{-1}=(2\ell)^{-1}\tilde\Sigma_\lambda\{(1-\lambda)^{-1}+(1-\bar\lambda)^{-1})$$

$$=(2\ell)^{-1}\tilde\Sigma_\lambda 1=(\ell-1)/(2\ell).$$

$$\eta(M^X_{3,0})((\rho_0-\rho_3)^2\rho_{-2})=\ell^{-1}\tilde\Sigma_\lambda(1+\lambda^3)=\ell^{-1}\tilde\Sigma_\lambda(1+\lambda)=(\ell-2)/\ell.$$

3.9 Lemma. *If $k\ge 0$ and if $\ell\ge 4$, then*

(1) $|\eta^*(\mathcal{M}^L_{4k+3}(\ell)|\ge(2\ell)^{k+1}$.

(2) $|\eta^*(\mathcal{M}^X_{4k+3}(\ell)|\ge(\ell/2)^{k+1}$.

(3) $\widetilde{ko}_{4k+3}(B\mathbb{Z}_\ell, \xi_0) \cap \ker \hat{A} = \mathcal{M}^L_{4k+3}(\ell)$.

(4) $ko_{4k+3}(B\mathbb{Z}_\ell, \xi_1) = \mathcal{M}^X_{4k+3}(\ell)$.

The first two assertions follow from Lemma 3.8 if $k = 0$ so we assume $k \geq 1$ henceforth. Let $\delta = (\rho_0 - \rho_3)\rho_{-2}$ and let $m = 4k+3$. Then

$$\eta(M^L_{m,j}(\ell))(\delta\rho) = \eta(M^L_{m-2,j}(\ell))(\rho) \text{ and,}$$

$$\eta(M^X_{m,j}(\ell))(\delta\rho) = \eta(M^X_{m-2,j}(\ell))(\rho).$$

Thus

$$\eta^*\mathcal{M}^L_{4k+1} \subset \delta^*\eta^*\mathcal{M}^L_{4k+3}, \text{ and } \eta^*\mathcal{M}^X_{4k+1} \subset \delta^*\eta^*\mathcal{M}^X_{4k+3}.$$

We use this equation and Lemma 3.6 to complete the proof of the first two assertions for $k \geq 1$ by computing:

$$(2\ell)^{k+1} \leq |\eta^*(\mathcal{M}^L_{4k+1}(\ell))| \leq |\eta^*(\mathcal{M}^L_{4k+3}(\ell))|$$

$$(\ell/2)^{k+1} \leq |\eta^*(\mathcal{M}^X_{4k+1}(\ell))| \leq |\eta^*(\mathcal{M}^X_{4k+3}(\ell))|$$

Assertions (3) and (4) follow from Table 3.7 except for $m = 8k+3$ and $\xi_=\xi_0$. In this case, we must squeeze out a single extra factor of 2. We use the refined eta invariant $\theta : M \mapsto \eta(M)(\rho_0 - \rho_{\ell/2}) \in \mathbb{R}/2\mathbb{Z}$. We complete the proof of the Lemma by constructing $M \in \ker(\eta^*)$ so that $\eta(M)(\rho_0 - \rho_{\ell/2}) \neq 0$ in $\mathbb{R}/2\mathbb{Z}$. If B^8 is the Bott manifold, then $\hat{A}(B^8) = 1$ and thus $\eta(M \times (B^8)^j)(\rho) = \eta(M)(\rho)$ by equation (2.11). Consequently it suffices to do this computation in the case $m = 3$. Let $M := \mathcal{B}L^3(\ell; 1, 3) = L^3(\ell; 1, 3) - 3L^3(\ell; 3, 3)$. We use Lemma 3.4 to see that $\eta(M)(\rho) = \eta(L^3(\ell; 1, 3))(\psi\rho)$. Since $\psi \in RU_0(\mathbb{Z}_\ell)^2$, we have $\eta(M)(\rho\psi) = 0$ in $\mathbb{R}/\mathbb{Z}$ for $\rho \in RU_0(\mathbb{Z}_\ell)$ and thus $\eta^*(M) = 0$. Let $s = \ell/2$. We compute:

$$\begin{aligned}\theta(M) =& \eta(M)(\rho_0 - \rho_s) = \eta(L^3(\ell; 1, 3))(\psi(\rho_0 - \rho_s)) \\ =& \ell^{-1}\tilde{\Sigma}_\lambda(1-\lambda)^2\lambda^{-1}(1-\lambda^s)\lambda^2/(1-\lambda)(1-\lambda^3) \\ =& \ell^{-1}\tilde{\Sigma}_\lambda\lambda(1-\lambda)(1-\lambda^{3s})/(1-\lambda^3) \\ =& \ell^{-1}\tilde{\Sigma}_\lambda\lambda(1-\lambda)(1+\lambda^3+\lambda^6+\ldots+\lambda^{3s-3}).\end{aligned}$$

We may replace $\tilde{\Sigma}_\lambda$ by Σ_λ since $1-\lambda=0$ if $\lambda=1$. We have $\ell^{-1}\Sigma_\lambda\lambda^j=\delta_{j,\ell}$ for $0<j<3s-3+2<2\ell$. We may expand $\ell=3s+t$ for $0\le t\le 2$. Since ℓ is not divisible by 3, we have $t=1$ or $t=2$. Thus exactly one term in this expansion is non-zero so we see $\theta(M)=\pm 1$. $\square$

As a scholium to the proof of Lemmas 3.6 and 3.9, we have the following Corollary which was noticed by Botvinnik, Gilkey, and Stolz [13]; see also Botvinnik and Gilkey [11].

3.10 Corollary. *Let* $[M]\in ko_m(B\mathbb{Z}_\ell)$.

(1) *If* $m\equiv 1$ *mod* 8, *then* $[M]=0$ *if and only if* $\eta(M)(\rho)=0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho\in RU(\mathbb{Z}_\ell)$ *and* $\hat{\mathbb{A}}[M]=0$.

(2) *If* $m\equiv 3$ *mod* 8, *then* $[M]=0$ *if and only if* $\eta(M)(\rho)=0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho\in RU_0(\mathbb{Z}_\ell)$ *and* $\eta(M)(\rho_0-\rho_{\ell/2})=0$ *in* $\mathbb{R}/2\mathbb{Z}$.

(3) *If* $m\equiv 5$ *mod* 8, *then* $[M]=0$ *if and only if* $\eta(M)(\rho)=0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho\in RU(\mathbb{Z}_\ell)$.

(4) *If* $m\equiv 7$ *mod* 8, *then* $[M]=0$ *if and only if* $\eta(M)(\rho)=0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho\in RU_0(\mathbb{Z}_\ell)$.

CHAPTER IV

TWISTED PRODUCTS

IV.1 Introduction

Let M be a compact connected Riemannian manifold of even dimension m which is not orientable. We assume the fundamental group $\mathbb{Z}_\ell$ is cyclic of order $\ell = 2^q \geq 2$. Let $\tilde{M}$ be the universal cover of M. Let the generator g_ℓ of $\pi_1(M)$ act on the universal cover $\tilde{M}$ by the deck group action $g_\ell : x \mapsto g_\ell \cdot x$. We assume that $\tilde{M}$ has a *spin* structure. We lift the action of g_ℓ to a morphism $\tilde{g}_\ell$ of the principal Pin^- bundle over $\tilde{M}$. Then $\tilde{g}_\ell^\ell$ covers the identity map of $\tilde{M}$ so $\tilde{g}_\ell^\ell = \pm 1$. If $\tilde{g}_\ell^\ell = +1$, then $\tilde{M}$ admits a pin^- structure p; if $\tilde{g}_\ell^\ell = -1$, then $\tilde{M}$ admits a pin^c structure p so that associated determinant representation $\det(p) = \rho_1$. Give $\tilde{M} \times \tilde{M}$ with the product *spin* structure. We define a fixed point free action of $\mathbb{Z}_{2\ell}$ on $\tilde{M} \times \tilde{M}$ by $g_{2\ell} : (x, y) \mapsto (g_\ell \cdot y, x)$; let

$$N = N(M) := \tilde{M} \times \tilde{M} / \mathbb{Z}_{2\ell}$$

be the resulting quotient manifold. Since the dimension m of M is even, the flip $(x, y) \mapsto (y, x)$ preserves the orientation of $\tilde{M} \times \tilde{M}$. Since g_ℓ reverses the orientation of $\tilde{M}$, $g_{2\ell}$ reverses the orientation of $\tilde{M} \times \tilde{M}$ so N is not orientable. We will show that N admits a suitable pin^c structure and express the eta invariant of N in terms of the eta invariant of M.

4.1 Example. Let $\tilde{M} = S^{2k}$ and let $g_2(x) = -x$ be the antipodal map. Then $M = \mathbb{RP}^{2k}$. The normal bundle of S^{2k} in $\mathbb{R}^{2k+1}$ is trivial. Consequently

$$T(S^{2k}) \oplus 1 = 1^{2k+1} \text{ and } T(\tilde{M} \times \tilde{M}) \oplus 1^2 = 1^{4k+2}.$$

If we let $g_4(z, w) = (w, z)$ define an action of $\mathbb{Z}_4$ on 1^2 and $g_4(\xi_1, \xi_2) = (-\xi_2, \xi_1)$ on 1^{4k+2}, then the decomposition given above is $\mathbb{Z}_4$ equivariant. Since we have $g_4(z, z) = (-z, z)$ and $g_4(z, -z) = (z, z)$, we see that

$$1^2/\mathbb{Z}_4 = 1 \oplus L$$

where $L = \rho_2(N)$ is the orientation line bundle of N. If we complexify and let $\zeta = \xi_1 + i\xi_2$, then $g_4(\zeta) = \sqrt{-1}\zeta$; this shows that

$$1^{4k+2}/\mathbb{Z}_4 = (2k+1)r(\rho_1)$$

is the underlying real 2-plane bundle defined by the complex representation ρ_1. We use the previous 3 equations to see that

$$T(N) \oplus 1 \oplus L = (2k+1)r(\rho_1).$$

We have $H^1(B\mathbb{Z}_4; \mathbb{Z}_2) = x \cdot \mathbb{Z}_2$ and $H^2(B\mathbb{Z}_4; \mathbb{Z}_2) = y \cdot \mathbb{Z}_2$ where $w(L) = 1 + x$ and $w(r(\rho_1)) = 1 + y$. Furthermore $x^2 = 0$. It now follows that

$$w_1(T(N)) = f^*(x) \text{ and } w_2(T(N)) = f^*y$$

where $f : N \mapsto B\mathbb{Z}_4$ is the canonical map. Consequently N does not admit a pin^- structure but does admit a pin^c structure p with $\det(p) = \rho_1$. Equivalently

$$[N] \in MSpin_{4k}(B\mathbb{Z}_4, L \oplus r(\rho_1)).$$

4.3 Example. Let $\tau = \rho_{a_1} \oplus ... \oplus \rho_{a_k}$ define the lens space $L^{2k-1}(\ell; \vec{a})$. We let $\mathbb{Z}_{2\ell}$ act on $S^{2k-1} \times S^{2k-1}$ by

$$g_{2\ell} : (x, y) \mapsto (\tau(g_\ell)y, x).$$

and using this action to define

$$M^{4k-1} := (S^{2k-1} \times S^{2k-1})/\mathbb{Z}_{2\ell}.$$

We then iterate the construction to let $\mathbb{Z}_{4\ell}$ act on $(S^{2k-1})^4$ by

$$g_{4\ell} : (x, y, z, w) \mapsto (g_{2\ell}(z, w), x, y) = (\tau(g_\ell)w, z, x, y)$$

and using this action to define

$$N = (S^{2k-1})^4/\mathbb{Z}_{4\ell}.$$

We define representations $\alpha : \mathbb{Z}_{4\ell} \to GL(4, \mathbb{R})$ and $\beta : \mathbb{Z}_{4\ell} \to GL(4k, \mathbb{C})$ by

$$\alpha(g_{4\ell}) : (t_1, t_2, t_3, t_4) \mapsto (t_4, t_3, t_1, t_2), \text{ and}$$

$$\beta(g_{4\ell}) : (x, y, z, w) \mapsto (\tau(g_\ell)w, z, x, y).$$

The equivariant decomposition $T(\tilde{M} \times \tilde{M}) \oplus 1^4 = 1^{8\ell}$ then gives

$$T(N) \oplus \alpha(N) = \beta(N).$$

Let

$$v_1 := (1, 1, 1, 1), \ v_2 := (1, 1, -1, -1),$$

$$v_3 := (1, -1, -1, 1), \ v_4 := (1, -1, 1, -1).$$

we see that $\alpha v_1 = v_1$, $\alpha v_2 = -v_2$, $\alpha v_3 = v_4$, $\alpha v_4 = -v_3$. Thus

$$\alpha = \rho_0 + \rho_{2k} + r(\rho_\ell) \text{ and } 1^4/\mathbb{Z}_{4\ell} = 1 \oplus L \oplus r(\rho_\ell(N))$$

where $L = \rho_{2\ell}(N)$ is the orientation line bundle over the non-orientable manifold N; $\rho_\ell(g_{4\ell}) = \sqrt{-1}$ defines the rotation with period 4. We note that $\beta(g_{4\ell})^4 = \tau(g_\ell) \otimes I_4$. Thus $\beta : \mathbb{Z}_{4\ell} \to U(4k)$ is fixed point free. We have that $\alpha(N)$ and $\beta(N)$ both have vanishing first and second Stiefel-Whitney classes; they both admit *spin* structures. Let $f : N \mapsto B\mathbb{Z}_{4\ell}$ classify the natural $\mathbb{Z}_{4\ell}$ structure on N. Then

$$[N] \in MSpin_{8k-4}(B\mathbb{Z}_{4\ell}, L),\ w_1(N) = f^*x,\ w_2(N) = 0.$$

If $\tau = \rho_a$ for a odd, then $\beta = \rho_a + \rho_{a+\ell} + \rho_{a+2\ell} + \rho_{a+3\ell}$; more generally, let $\tilde{\tau}$ be any lift if τ from a fixed point free representation of $\mathbb{Z}_\ell$ to a fixed point free representation of $\mathbb{Z}_{4\ell}$. Then

$$\beta = \tilde{\tau} + (\tilde{\tau} \otimes \rho_\ell) + (\tilde{\tau} \otimes \rho_{2\ell}) + (\tilde{\tau} \otimes \rho_{3\ell}).$$

This will play a crucial role in what follows.

IV.2 Spinor Bundles

$Clif^-(\mathbb{R}^m)$ be the real Clifford algebra and let Δ_m be the spin representation for m even; Clifford multiplication defines a natural map

$$c_m : \mathbb{R}^m \otimes_{\mathbb{R}} \Delta_m \mapsto \Delta_m$$

so that $c_m(\xi)^2 = -|\xi|^2 1$. Let $\{e_1, ..., e_m\}$ be an oriented orthonormal basis for $\mathbb{R}^m$. We define the normalized orientation class

$$\omega_m := (\sqrt{-1})^{m/2} e_1 * ... * e_m \in Clif^-(\mathbb{R}^m).$$

The normalization is chosen so that $\omega_m^2 = 1$. We define the associated Cliford multiplication

$$\check{c}_m(\xi) = \sqrt{-1} c_m(\omega_m) c_m(\xi);$$

since $c_m(\omega_m)$ anti-commutes with $c_m(\xi)$, we have that $\check{c}_m(\xi)^2 = -|\xi|^2$ so $\check{c}_m$ also defines a representation of $Clif^-(\mathbb{R}^m)$ on Δ_m. Since m is even,

$$\check{c}_m(\omega_m) = c_m(\omega_m) \text{ and } c_m(\xi) = -\sqrt{-1}\check{c}_m(\omega_m)\check{c}_m(\xi)$$

so apart from a sign convention, the roles of the two representations are symmetric. If $\chi : Pin(m) \to \mathbb{Z}_2$ is the orientation representation §II.4, then

$$c_m(\omega_m)c_m(g) = \chi(g)c_m(g)c_m(\omega_m).$$

Furthermore, we have $c_m(g) = \check{c}_m(g)$ if $g \in Spin(m)$. Let

$$\Psi(g) : \xi \to \chi(g)g * \xi * g^{-1}$$

define the canonical representation from $Pin^-(m)$ to $SO(m)$. The following diagram:

$$\begin{array}{ccc} \mathbb{R}^m \otimes \Delta_m & \xrightarrow{\check{c}_m} & \Delta_m \\ (\Psi \otimes c_m)(g) \downarrow & \circ & c_m(g) \downarrow \\ \mathbb{R}^m \otimes \Delta_m & \xrightarrow{\check{c}_m} & \Delta_m \end{array} \tag{4.3}$$

commutes because

$$\begin{aligned} &\check{c}_m(\chi(g)g * \xi * g^{-1})c_m(g) = \sqrt{-1}c_m(\omega_m)c_m(\chi(g)g * \xi * g^{-1})c_m(g) \\ &=\sqrt{-1}c_m(\omega_m)c_m(\chi(g)g * \xi) = \sqrt{-1}c_m(g)c_m(\omega_m * \xi) = c_m(g)\check{c}_m(\xi). \end{aligned}$$

We use the representation c_m to define the bundle Δ_m of spinors and the representation $\check{c}_m$ to define the leading symbol of the Dirac operator Q on the spinor bundle.

We now describe a general construction. Let $\tilde{X}$ be a simply connected *spin* manifold of even dimension ν. If g is an isometry of $\tilde{X}$, lift g to an action $\tilde{g}$ on the principal *pin* bundle of $\tilde{X}$ and let $\mathcal{S}_g$ be the associated action on the bundle $\Delta_\nu(\tilde{X})$ defined by c_ν which covers the map g. Then we have the relations

$$\mathcal{S}_g Q = Q\mathcal{S}_g,\ \mathcal{S}_g c_\nu(\omega_\nu) = \chi(g)c_\nu(\omega_\nu)\mathcal{S}_g,\ c_\nu(\omega_\nu)Q = -Qc_\nu(\omega_\nu). \tag{4.4}$$

If $\xi \oplus \tilde{\xi} \in \mathbb{R}^m \oplus \mathbb{R}^m = \mathbb{R}^{2m}$, we define

$$\begin{aligned} &\check{c}_{2m}(\xi \oplus \tilde{\xi}) := c_m(\xi) \otimes 1 + c_m(\omega_m) \otimes c_m(\tilde{\xi}), \\ &\check{c}_{2m}(\omega_{2m}) = c_m(\omega_m) \otimes c_m(\omega_m), \\ &c_{2m}(\xi \oplus \tilde{\xi}) := -\sqrt{-1}\check{c}_{2m}(\omega_{2m})\check{c}_{2m}(\xi \oplus \tilde{\xi}). \end{aligned} \tag{4.5}$$

Since m is even, $c_m(\omega_m)$ anti-commutes with $c_m(\xi)$. Consequently

$$\check{c}_{2m}(\xi \oplus \tilde{\xi})^2 = -(|\xi|^2 + |\tilde{\xi}|^2).$$

Thus $\check{c}_{2m}$ and c_{2m} define representations of the Cliford algebra; for dimensional reasons, $\check{c}_{2m}$ and c_{2m} are both isomorphic to the unique irreducible representation of $Clif^-(\mathbb{R}^{2m})$; they agree on $Spin(2m)$. Thus we may identify the *spin* bundle $\Delta_{2m}(\tilde{M} \times \tilde{M})$ and the Dirac operator P on $C^\infty(\Delta_{2m}(\tilde{M} \times \tilde{M}))$ with the corresponding objects over $\tilde{M}$:

$$\begin{aligned} &\Delta_{2m}(\tilde{M} \times \tilde{M}) = \Delta_m(\tilde{M}) \otimes \Delta_m(\tilde{M}) \\ &P = Q \otimes 1 + c_m(\omega_m) \otimes Q. \end{aligned} \tag{4.6}$$

Let $\mathcal{S}_\ell$ and $\mathcal{S}_{2\ell}$ denote the actions of $\tilde{g}_\ell$ and $\tilde{g}_{2\ell}$ on the bundles $\Delta_m(\tilde{M})$ and $\Delta_{2m}(\tilde{M} \times \tilde{M})$ as was discussed above. Decompose $\Delta_m(\tilde{M}) = \Delta_m^+(\tilde{M}) \oplus \Delta_m^-(\tilde{M})$ into the ± 1 eigenspaces of $c_m(\omega_m)$ where ω_m is the orientation of $\tilde{M}$. Let $F(x, y) = (y, x)$ interchange the two factors of $\tilde{M} \times \tilde{M}$. Let

$$\begin{aligned} &\alpha(v_+ \otimes w_+)(x, y) := w_+(y, x) \otimes v_+(y, x), \\ &\alpha(v_+ \otimes w_-)(x, y) := w_-(y, x) \otimes v_+(y, x), \\ &\alpha(v_- \otimes w_+)(x, y) := w_+(y, x) \otimes v_-(y, x), \\ &\alpha(v_- \otimes w_-)(x, y) := - w_-(y, x) \otimes v_-(y, x); \end{aligned} \tag{4.7}$$

The definition of α is motivated by the corresponding action on the exterior algebra; we decompose

$$\Lambda(\tilde{M} \times \tilde{M}) = \Lambda(\tilde{M}) \otimes \Lambda(\tilde{M}) \text{ and } F^*(\phi_p \wedge \psi_q) = (-1)^{pq}\psi_q \wedge \phi_p$$

so F^* interchanges the factors and introduces a minus sign if both forms are of odd degree. Let $\mathcal{S}_F$ be the action of F on $\Delta_{2m}(\tilde{M} \times \tilde{M})$.

4.8 Lemma. *Let M be a compact connected manifold of even dimension m with fundamental group $\mathbb{Z}_\ell$. Assume the universal cover $\tilde{M}$ of M is spin.*

a) *We have $\mathcal{S}_F = (\sqrt{-1})^{m/2}\alpha$.*

b) *We have $\mathcal{S}_{2\ell} = (\mathcal{S}_\ell \otimes c(\omega_m)) \circ \mathcal{S}_F$.*

c) *If $\ell = 2$, then $[N] \in MSpin_{2m}(B\mathbb{Z}_4, L \oplus r(\rho_1))$.*

d) *If $\ell \geq 4$, then $[N] \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, L)$.*

Proof. We use equation (4.3) to see that $\check{c}_{2m}(\tilde{\xi}, \xi)\mathcal{S}_F = \mathcal{S}_F \check{c}_{2m}(\xi, \tilde{\xi})$. If we can show that

$$c_{2m}(\tilde{\xi}, \xi)\alpha = \alpha\check{c}_{2m}(\xi, \tilde{\xi}), \tag{4.9}$$

we will have that $\alpha^{-1}\mathcal{S}_F$ commutes with $\check{c}_{2m}(\xi, \tilde{\xi})$. Since the representation $\check{c}_{2m}$ is irreducible, this will imply $\alpha^{-1}\mathcal{S}_F$ is scalar so $\mathcal{S}_F = \epsilon\alpha$. Let $\{e_i\}$ be an oriented orthonormal basis for $\mathbb{R}^m$. The lift of F from $SO(2m)$ to $Spin(2m)$ is given by

$$\tilde{F} := 2^{-m/2}\Pi_{1\leq i\leq m}(e_i - \tilde{e}_i)$$

since we must reflect in these hyperplanes to interchange the coordinates. Consequently $\tilde{F}^2 = (-1)^{m/2}$ so $\mathcal{S}_F^2 = (-1)^2 = \epsilon^2\alpha^2 = \epsilon^2$. Thus $\epsilon = \pm 1$ if $m \equiv 0$ (4) and $\epsilon = \pm\sqrt{-1}$ if $m \equiv 2$ (4); we can adjust the sign of ϵ by changing the sign of the lift $\tilde{F}$ which is chosen; (a) will then follow.

The argument given above shows that to prove (a), it suffices to establish equation (4.9). Note $c_m(\xi) : \Delta_m^\pm \to \Delta_m^\mp$. Let $v_\pm \in \Delta_m^\pm(x)$ and $w_\pm \in \Delta_m^\pm(y)$ for $x \in \tilde{M}$ and $y \in \tilde{M}$. We compute:

$$\alpha\check{c}_{2m}(\xi, 0)(v_+ \otimes w_+) = \alpha(c_m(\xi)v_+ \otimes w_+) = w_+ \otimes c_m(\xi)v_+$$

$$\check{c}_{2m}(0, \xi)\alpha(v_+ \otimes w_+) = \check{c}_{2m}(0, \xi)(w_+ \otimes v_+) = w_+ \otimes c_m(\xi)v_+$$

$$\alpha\check{c}_{2m}(\xi,0)(v_-\otimes w_+) = \alpha(c_m(\xi)v_-\otimes w_+) = w_+\otimes c_m(\xi)v_-$$

$$\check{c}_{2m}(0,\xi)\alpha(v_-\otimes w_+) = \check{c}_{2m}(0,\xi)(w_+\otimes v_-) = w_+\otimes c_m(\xi)v_-$$

$$\alpha\check{c}_{2m}(\xi,0)(v_+\otimes w_-) = \alpha(c_m(\xi)v_+\otimes w_-) = -w_-\otimes c_m(\xi)v_+$$

$$c_{2m}(0,\xi)\alpha(v_+\otimes w_-) = \check{c}_{2m}(0,\xi)(w_-\otimes v_+) = -w_-\otimes c_m(\xi)v_+$$

$$\alpha\check{c}_{2m}(\xi,0)(v_-\otimes w_-) = \alpha(c_m(\xi)v_-\otimes w_-) = w_-\otimes c_m(\xi)v_-$$

$$\check{c}_{2m}(0,\xi)\alpha(v_-\otimes w_-) = -\check{c}_{2m}(0,\xi)(w_-\otimes v_-) = (-1)^2 w_-\otimes c_m(\xi)v_-.$$

This shows $\check{c}_{2m}(\xi,0)\alpha = \alpha\check{c}_{2m}(0,\xi)$. Since $\alpha^2 = 1$, $\alpha\check{c}_{2m}(\xi,0) = \check{c}_{2m}(0,\xi)\alpha$ as well. This establishes equation (4.9) and completes the proof of the first assertion. Let $h:(x,y)\to(g_\ell\cdot x,y)$. Lift h to a *pin* morphism $\tilde{h}$. Then $\check{c}_{2m}(\tilde{h}) = \check{c}_m(\tilde{g}_\ell)\otimes 1$. Since h reverses the orientation,

$$\begin{aligned}\mathcal{S}_h =& c_{2m}(\tilde{h}) = -\sqrt{-1}\check{c}_{2m}(\omega_{2m})\check{c}_{2m}(\tilde{h})\\ =& -\sqrt{-1}(\check{c}_m(\omega_m)\otimes\check{c}_m(\omega_m))(\check{c}_m(\tilde{g}_\ell)\otimes 1)\\ =& \mathcal{S}_\ell\otimes c_m(\omega_m).\end{aligned}$$

Assertion (b) now follows from assertion (a).

We have that $g_{2\ell}^4 = g_\ell^2\times g_\ell^2$ so $\tilde{g}_{2\ell}^4 = \pm\tilde{g}_\ell^2\times\tilde{g}_\ell^2$ where we use the canonical embeddings

$$SO(m)\times SO(m)\subset SO(2m) \text{ and } Spin(m)\times Spin(m)\subset Spin(2m).$$

Consequently $\mathcal{S}_{2\ell}^4 = \epsilon\mathcal{S}_\ell^2\otimes\mathcal{S}_\ell^2$ where $\epsilon = \epsilon(m) = \pm 1$ depends only on the dimension m. We have $\mathcal{S}_{2\ell}^{2\ell} = \epsilon^{\ell/2}\mathcal{S}_\ell^\ell\otimes\mathcal{S}_\ell^\ell = \epsilon^{\ell/2}$ since $\mathcal{S}_\ell^\ell = \pm 1$. Thus $\mathcal{S}_{2\ell}^{2\ell} = +1$ if $\ell\geq 4$ which proves assertion d). If $\ell = 2$, we have $\mathcal{S}_{2\ell}^{2\ell} = \epsilon(m)$. We use Example 4.1 to see that $\epsilon(m) = -1$; assertion c) now follows. $\square$

CHAPTER V

THE ETA INVARIANT OF TWISTED PRODUCTS

V.1 Introduction

In this section, we express the eta invariant of the twisted product N defined in the previous section in terms of the eta invariant of M.

V.2 Equivariant Eta Invariant

We introduce the equivariant eta function to encode the information contained in the ordinary eta invariant. The map which sends ρ to $Tr(\rho)$ embeds the group representation ring $RU(\mathbb{Z}_\ell)$ in $L^2(\mathbb{Z}_\ell)$; by the orthogonality relations, $\{\rho_s = Tr(\rho_s)\}$ is an orthonormal basis for $L^2(\mathbb{Z}_\ell)$ where $s \in \mathbb{Z}_\ell^*$, i.e. $0 \leq s \leq \ell - 1$.

We adopt the notation of §4. Let M be a closed connected non-orientable Riemannian manifold of even dimension m with fundamental group $\pi_1(M) = \mathbb{Z}_\ell$. Assume the universal cover $\tilde{M}$ of M admits a *spin* structure. Then M admits a pin^c structure s whose associated determinant line bundle is given by ρ_b. If b is even, M admits a pin^- structure. Define

$$\tilde{\eta}(M) := \Sigma_s \eta(M)(\rho_s) \cdot \rho_s \in L^2(\mathbb{Z}_\ell). \tag{5.1}$$

Expanding $\tilde{\eta}(M)$ in terms of the orthonormal basis provided by the ρ_s then permits us to recover $\eta(M)(\rho_s)$.

We define a representation $\tau : \mathbb{Z}_{2\ell} \to U(2)$ by setting

$$(5.2)\qquad \begin{aligned} \tau(g_4) =&(\sqrt{-1})^{m/2} e^{-2\pi\sqrt{-1}b/2\ell} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \text{ and} \\ \tau(g_{2\ell}) =&(\sqrt{-1})^{m/2} e^{-2\pi\sqrt{-1}b/2\ell} \begin{pmatrix} e^{2\pi\sqrt{-1}/8} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/8} \end{pmatrix} \text{ if } \ell > 2. \end{aligned}$$

Let $r(g_{2\ell}) = g_\ell$ define a surjective map from $\mathbb{Z}_{2\ell}$ to $\mathbb{Z}_\ell$. Pull-back defines a ring homomorphism $r^* : L^2(\mathbb{Z}_\ell) \to L^2(\mathbb{Z}_{2\ell})$; $r^*(\rho_s) = \rho_{2s}$.

5.3 Theorem. *Let M and N be as above. Then*

$$\tilde{\eta}(N) = r^*(\tilde{\eta}(M)) \cdot Tr(\tau).$$

5.4 Remark. We use equations (5.1) and (5.2) to see that Theorem 5.3 is equivalent to the assertions:

(1) If $\ell = 2$ and if $u = 2s - b + m/2$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$.

(2) If $\ell = 2$ and if $u = 2s - b + 1 + m/2$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$.

(3) If $\ell \geq 4$ and if $u = 2s - b + m/2 + \ell/4$, then
$\eta(N)(\rho_u) = \eta(M)(\rho_s) + \eta(M)(\rho_{s+\ell/4})$.

(4) If $\ell \geq 4$ and if $u = 2s - b + m/2 + \ell/4 + 1$, then $\eta(N)(\rho_u) = 0$.

Proof. We work equivariantly to compute the eta invariant of N and of M. Decompose

$$L^2(\Delta_m(\tilde{M})) = \oplus_{\lambda \in \mathbb{R}} E(\lambda, \tilde{M})$$

into the eigenspaces of the Dirac operator on the universal cover $\tilde{M}$. Since $\mathbb{Z}_\ell$ acts by spinor isometries which commute with the Dirac operator, the eigenspaces are representation spaces for $\mathbb{Z}_\ell$ and we may further decompose each eigenspace

$E(\lambda, \tilde{M}) = \oplus_{0 \leq s < \ell} E_s(\lambda, \tilde{M})$ for

$E_s(\lambda, \tilde{M}) := \{\phi \in C^\infty(\Delta_m(\tilde{M})) : Q\phi = \lambda\phi, e^{2\pi\sqrt{-1}b/2\ell} \mathcal{S}_\ell \phi = e^{2\pi\sqrt{-1}s/\ell} \phi\}$

where Q is Dirac operator on the spinor bundle defined previously. We identify spinors on M with equivariant sections to $\Delta_m(\tilde{M})$ so

$$E(\lambda, Q_M) = E_0(\lambda, \tilde{M}).$$

Similarly, sections to $\Delta_m(M) \otimes \rho_t(M)$ may be identified with spinors on M which transform appropriately with respect to the group action;

$$E(\lambda, Q_M \otimes \rho_s) = E_s(\lambda, \tilde{M})$$

is the eigenspace for the Dirac operator with coefficients in the representation ρ_s on M. Since there are no harmonic spinors,

$$\eta(M)(\rho_s) = \tfrac{1}{2}\{\Sigma_{\lambda \neq 0} \dim E_s(\lambda, \tilde{M}) \mathrm{sign}(\lambda)|\lambda|^{-z}\}|_{z=0}.$$

It is now immediate that

$$\tilde{\eta}(M)(g) = \tfrac{1}{2}\{\Sigma_{\lambda \neq 0} Tr(\mathcal{S}_g \text{ on } E(\lambda, \tilde{M}))\mathrm{sign}(\lambda)|\lambda|^{-z}\}|_{z=0}. \tag{5.5}$$

Let $\mathcal{S}(h)$ denote the action of $h \in \mathbb{Z}_{2\ell}$ on $\Delta_{2m}(N)$ and let $\mathcal{S}(h^2)$ be the corresponding action of $h^2 \in \mathbb{Z}_\ell$ on $\Delta_m(N)$. Let

$$\mathcal{T}(\lambda, \cdot) := Tr(\mathcal{S}(\cdot) \text{ on } E(\lambda, \cdot)) - Tr(\mathcal{S}(\cdot) \text{ on } E(-\lambda, \cdot))$$

be the super or $\mathbb{Z}_2$ graded trace. We will show that

$$\mathcal{T}(\sqrt{2}\lambda, P) = (\sqrt{-1})^{m/2}(e^{2\pi\sqrt{-1}/8} + e^{-2\pi\sqrt{-1}/8}) = \mathcal{T}(\lambda, Q) \tag{5.6}$$

where P and Q are related in equation (4.6). Taking into account the normalizing factor of $e^{2\pi\sqrt{-1}b/2\ell}$ in defining the pin^c structure on M, we use equation(5.5) to complete the proof.

We establish equation(5.6) holds by giving an equivariant spectral resolution for the Dirac operator on N in terms of the equivariant spectral resolution on M. We change notation slightly. Let $L^2(\Delta_m(\tilde{M})) = \oplus_{i,s}\phi_{i,s} \cdot \mathbb{C}$ be an equivariant spectral resolution of the Dirac operator on $\tilde{M}$ where

$$Q\phi_{i,s} = \mu_{i,s}\phi_{i,s} \text{ and } \mathcal{S}_\ell e^{2\pi\sqrt{-1}b/2\ell}\phi_{i,s} = e^{2\pi\sqrt{-1}s/\ell}\phi_{i,s}.$$

We have assumed there are no harmonic spinors. Since Q anti-commutes with $c(\omega_m)$, $Qc(\omega_m)\phi_{i,s} = -\mu_{i,s}c(\omega_m)\phi_{i,s}$. We restrict to $\mu_{i,s} > 0$ and $\mu_{j,t} > 0$ and decompose

$$\begin{aligned} &L^2(\Delta_{2m}(\tilde{N})) = \oplus_{i,s,j,t}E(i,s,j,t) \text{ for} \\ &E(i,s,j,t) := \operatorname{span}_{\mathbb{C}}\{\phi_{i,s}\otimes\phi_{j,t}, c_m(\omega_m)\phi_{i,s}\otimes\phi_{j,t}, \\ &\qquad \phi_{i,s}\otimes c_m(\omega_m)\phi_{j,t}, c_m(\omega_m)\phi_{i,s}\otimes c_m(\omega_m)\phi_{j,t}\}. \end{aligned}$$

These spaces are invariant under the action of the Dirac operator Q. We use Lemma 4.8 to see that the spaces $E(i,s,i,s)$ and $E(i,s,j,t)\oplus E(j,t,i,s)$ are invariant under $\mathbb{Z}_{2\ell}$. We will show that the spaces for $(i,s)\neq(j,t)$ contribute nothing to the super trace and will study the spaces $E(i,s,i,s)$ to complete the proof.

Let $\phi = \phi_{i,s}$ and let $\chi = c_m(\omega_m)$. Let

$$E = E(i,s,i,s) = \operatorname{span}_{\mathbb{C}}\{\phi\otimes\phi, \chi\phi\otimes\phi, \phi\otimes\chi\phi, \chi\phi\otimes\chi\phi\}.$$

This is not a particularly convenient basis for E. Define:

$$\epsilon_\pm := \pm\sqrt{2}-1,\ \ \Phi_\pm := \phi\otimes\phi + \epsilon_\pm\chi\phi\otimes\phi,\ \ \Psi_\pm := \chi\phi\otimes\chi\phi + \epsilon_\mp\phi\otimes\chi\phi.$$

Relative to this basis, the operator P defined in equation (4.6) is diagonal:

$$\begin{aligned} P\Phi_\pm &= \lambda\{(1+\epsilon_\pm)\phi\otimes\phi + (1-\epsilon_\pm)\chi\phi\otimes\phi\} = \pm\lambda\sqrt{2}\Phi_\pm \\ P\Psi_\pm &= -\lambda\{(1+\epsilon_\mp)\chi\phi\otimes\chi\phi + (1-\epsilon_\mp)\phi\otimes\chi\phi\} = \pm\lambda\sqrt{2}\Psi_\pm. \end{aligned}$$

Let $F(u \otimes v) = v \otimes u$ interchange the two factors. We compute:

$$
\begin{aligned}
4\alpha &= F \circ \{(1+\chi) \otimes (1+\chi) + (1+\chi) \otimes (1-\chi) \\
&\quad +(1-\chi) \otimes (1+\chi) - (1-\chi) \otimes (1-\chi)\} \\
&= 2F \circ (1 \otimes 1 + \chi \otimes 1 + 1 \otimes \chi - \chi \otimes \chi)
\end{aligned}
$$

$$
\begin{aligned}
2\alpha\phi \otimes \phi &= F(\phi \otimes \phi + \chi\phi \otimes \phi + \phi \otimes \chi\phi - \chi\phi \otimes \chi\phi) \\
&= \phi \otimes \phi + \chi\phi \otimes \phi + \phi \otimes \chi\phi - \chi\phi \otimes \chi\phi
\end{aligned}
$$

$$
\begin{aligned}
2\alpha(\chi\phi \otimes \phi) &= F(\chi\phi \otimes \phi + \phi \otimes \phi + \chi\phi \otimes \chi\phi - \phi \otimes \chi\phi) \\
&= \phi \otimes \phi - \chi\phi \otimes \phi + \phi \otimes \chi\phi + \chi\phi \otimes \chi\phi
\end{aligned}
$$

$$
\begin{aligned}
2\alpha(\phi \otimes \chi\phi) &= F(\phi \otimes \chi\phi + \chi\phi \otimes \chi\phi + \phi \otimes \phi - \chi\phi \otimes \phi) \\
&= \phi \otimes \phi + \chi\phi \otimes \phi - \phi \otimes \chi\phi + \chi\phi \otimes \chi\phi
\end{aligned}
$$

$$
\begin{aligned}
2\alpha(\chi\phi \otimes \chi\phi) &= F(\chi\phi \otimes \chi\phi + \phi \otimes \chi\phi + \chi\phi \otimes \phi - \phi \otimes \phi) \\
&= -\phi \otimes \phi + \chi\phi \otimes \phi + \phi \otimes \chi\phi + \chi\phi \otimes \chi\phi
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
2\alpha\Phi_\pm &= \phi \otimes \phi + \chi\phi \otimes \phi + \phi \otimes \chi\phi - \chi\phi \otimes \chi\phi \\
&\quad +\epsilon_\pm\phi \otimes \phi - \epsilon_\pm\chi\phi \otimes \phi + \epsilon_\pm\phi \otimes \chi\phi + \epsilon_\pm\chi\phi \otimes \chi\phi \\
&= \pm\sqrt{2}\Phi_\pm + (\pm\sqrt{2} - 2)\Psi_\pm
\end{aligned}
$$

$$
\begin{aligned}
2\alpha\Psi_\pm &= -\phi \otimes \phi + \chi\phi \otimes \phi + \phi \otimes \chi\phi + \chi\phi \otimes \chi\phi \\
&\quad +\epsilon_\mp\phi \otimes \phi + \epsilon_\mp\chi\phi \otimes \phi - \epsilon_\mp\phi \otimes \chi\phi + \epsilon_\mp\chi\phi \otimes \chi\phi \\
&= (\mp\sqrt{2} - 2)\Phi_\pm \mp \sqrt{2}\Psi_\pm.
\end{aligned}
$$

Recall that $\mathcal{S}_\ell\phi = \mu\phi$ and that $\mathcal{S}_\ell\chi\phi = -\mu\chi\phi$ for $\mu = e^{2\pi\sqrt{-1}s/\ell}e^{-2\pi\sqrt{-1}b/2\ell}$. Consequently

$$
\begin{aligned}
(\mathcal{S}_\ell \otimes \chi)\Phi_\pm &= \mu\phi \otimes \chi\phi - \mu\epsilon_\pm\chi\phi \otimes \chi\phi = -\mu(\pm\sqrt{2} - 1)\Psi_\pm \\
(\mathcal{S}_\ell \otimes \chi)\Psi_\pm &= -\mu\phi \otimes \chi\phi + \mu\epsilon_\mp\phi \otimes \phi = \mu(\mp\sqrt{2} - 1)\Phi_\pm.
\end{aligned}
$$

We summarize these computations. Relative to the basis $\{\Phi_+, \Psi_+, \Phi_-, \Psi_-\}$ for $E_{i,s,i,s}$ we have that

$$P = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$2\alpha = \begin{pmatrix} \sqrt{2} & -\sqrt{2}-2 & 0 & 0 \\ \sqrt{2}-2 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2}-2 \\ 0 & 0 & -\sqrt{2}-2 & \sqrt{2} \end{pmatrix},$$

$$\mathcal{S}_\ell \otimes \chi = \mu \begin{pmatrix} 0 & -\sqrt{2}-1 & 0 & 0 \\ -\sqrt{2}+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}-1 \\ 0 & 0 & \sqrt{2}+1 & 0 \end{pmatrix},$$

$$2\mathcal{S}_{2\ell} = \mu(\sqrt{-1})^{m/2} \begin{pmatrix} \sqrt{2} & 2+\sqrt{2} & 0 & 0 \\ \sqrt{2}-2 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 2-\sqrt{2} \\ 0 & 0 & -\sqrt{2}-2 & -\sqrt{2} \end{pmatrix}$$

The eigenvalues of the matrices

$$\tfrac{1}{2}\begin{pmatrix} \sqrt{2} & 2+\sqrt{2} \\ \sqrt{2}-2 & \sqrt{2} \end{pmatrix} \text{ and } \tfrac{1}{2}\begin{pmatrix} -\sqrt{2} & 2-\sqrt{2} \\ -\sqrt{2}-2 & -\sqrt{2} \end{pmatrix}$$

are $\{e^{2\pi\sqrt{-1}/8},\, e^{-2\pi\sqrt{-1}/8}\}$ and $\{-e^{2\pi\sqrt{-1}/8}, -e^{-2\pi\sqrt{-1}/8}\}$ respectively.

We summarize the results of our computation. Let the action of g_ℓ on the $+\lambda$ eigenspace of Q which is generated by ϕ be given by $e^{2\pi\sqrt{-1}s/\ell}$ and on the $-\lambda$ eigenspace which is generated by $\chi\phi$ be given by $e^{-2\pi\sqrt{-1}s/\ell}$. We set

$$\pm\mu = \pm e^{2\pi\sqrt{-1}s/\ell} e^{-2\pi\sqrt{-1}b/2\ell} (\sqrt{-1})^{-m/2}.$$

Then after a suitable change of basis, we see the action of $g_{2\ell}$ on the appropriate $\pm\sqrt{2}\lambda$ eigenspaces of P is given by $\pm\mu \cdot \mathrm{diag}(e^{2\pi\sqrt{-1}/8}, e^{-2\pi\sqrt{-1}/8})$. This leads to the formula expressed in equation (5.6); we complete the proof of Theorem 5.3 by showing that the remaining eigenspaces make no contribution to the equivariant eta function.

Let $E := E(i,s,j,t) \oplus E(j,t,i,s)$ for $(i,s) \neq (j,t)$ define an 8 dimensional subspace of $L^2(\Delta_{2m}(\tilde{N}))$ which is invariant under the Dirac operator P and also under the action $\mathcal{S}_{2\ell}$. Let $\theta = (\lambda_i^2 + \lambda_j^2)$. Then $P = Q \otimes 1 + \chi \otimes Q$ so $P^2 = Q^2 \otimes 1 + 1 \otimes Q^2 = \theta^2$ on E. Let $E^\pm$ be the $\pm\theta$ eigensections for P. We showed previously that we had $\mathcal{S}_{2\ell}^4 = -\mathcal{S}_\ell^2 \otimes \mathcal{S}_\ell^2$. We have that $\mathcal{S}_{2\ell}$ commutes with P; let $\mathcal{S}_{2\ell}^\pm$ be the restriction of $\mathcal{S}_{2\ell}$ to $E^\pm$. We will show

$$Tr(\mathcal{S}_{2\ell}^\pm) = 0 \text{ and } Tr((\mathcal{S}_{2\ell}^\pm)^2) = 0. \tag{5.7}$$

Let $\mu = e^{2\pi\sqrt{-1}(s+t)/2\ell}$; $\mathcal{S}_{2\ell}^4 = -\mu^4$. Thus the eigenvalues of $\mu^{-1}\mathcal{S}_{2\ell}$ are primitive 8^{th} roots of unity. Let $\varrho = e^{2\pi\sqrt{-1}/8}$. Since $Tr(\mathcal{S}_{2\ell}^+) = 0$, the eigenvalues of $\mathcal{S}_{2\ell}^+$ must be

(1) $\mu(\varrho, -\varrho, \varrho^3, -\varrho^3)$,
(2) $\mu(\varrho, \varrho, -\varrho, -\varrho)$,
(3) $\mu(\varrho^3, \varrho^3, -\varrho^3, -\varrho^3)$;

other possible combinations of primitive 8^{th} roots of unity will not be trace free. Since $Tr((\mathcal{S}_{2\ell}^+)^2) = 0$, possibilities (2) and(3) are ruled out. Thus the eigenvalues of $\mathcal{S}_{2\ell}^+$ are $\mu(\varrho, -\varrho, \varrho^3, -\varrho^3)$. Similarly the eigenvalues of $\mathcal{S}_{2\ell}^-$ are also of the form $\mu(\varrho, -\varrho, \varrho^3, -\varrho^3)$. In particular, the eigenvalues of $\mathcal{S}_{2\ell}^+$ agree with the eigenvalues of $\mathcal{S}_{2\ell}^-$. Multiplying by a normalizing root of unity does not change this equality of eigenvalues. This shows that contribution made by the positive eigenvalue θ cancels the contribution made by the negative eigenvalue $-\theta$ in the equivariant eta function which will complete the proof.

We complete the proof of Theorem 5.3 by establishing equation (5.7). The 4^{th} root of unity $(\sqrt{-1})^{m/2}$ does not affect the computations so we may ignore it. We have assumed there are no zero eigenvalues so $\mu_{i,s} \neq 0$ and $\mu_{j,t} \neq 0$.

$$\epsilon_\pm := (\pm\theta - \mu_{i,s})/\mu_{j,t} \text{ and } \tilde{\epsilon}_\pm := (\pm\theta - \mu_{j,t})/\mu_{i,s},$$

$$\Phi_\pm := \phi_{i,s} \otimes \phi_{j,t} + \epsilon_\pm \chi \phi_{i,s} \otimes \phi_{j,t},$$

$$\Psi_\pm := \chi \phi_{i,s} \otimes \chi \phi_{j,t} + \epsilon_\mp \phi i, s \otimes \chi \phi_{j,t},$$

$$\tilde{\Phi}_\pm := \phi_{j,t} \otimes \phi_{i,s} + \tilde{\epsilon}_\pm \chi \phi_{j,t} \otimes \phi_{i,s},$$

$$\tilde{\Psi}_\pm = \chi \phi_{j,t} \otimes \chi \phi_{i,s} + \tilde{\epsilon}_\mp \phi_{j,t} \otimes \chi \phi_{i,s}.$$

We then have $\{\Phi_\pm, \Psi_\pm, \tilde{\Phi}_\pm, \tilde{\Psi}_\pm\}$ is a basis for $E^\pm$. Relative to this basis, we have

$$2\alpha = \begin{pmatrix} 0 & 0 & 1+\tilde{\epsilon}_\pm & \tilde{\epsilon}_\mp - 1 \\ 0 & 0 & \tilde{\epsilon}_\pm - 1 & \tilde{\epsilon}_\mp + 1 \\ 1+\epsilon_\pm & \epsilon_{\mp-1} & 0 & 0 \\ \epsilon_\pm - 1 & \epsilon_{\mp+1} & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\mathcal{S}_\ell \otimes \chi = \begin{pmatrix} 0 & \epsilon_\mp e^{2\pi\sqrt{-1}s/\ell} & 0 & 0 \\ -\epsilon_\pm e^{2\pi\sqrt{-1}s/\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_\mp e^{2\pi\sqrt{-1}t/\ell} \\ 0 & 0 & -\tilde{\epsilon}_\pm e^{2\pi\sqrt{-1}t/\ell} & 0 \end{pmatrix}.$$

Let $A_\pm = (\mathcal{S}_\ell \otimes \chi) \cdot 2\alpha|_{E_\pm}$. We use the block structure given above to see $Tr(A) = 0$. Furthermore $B_\pm := e^{-2\pi\sqrt{-1}(t+s)/\ell} A_\pm^2$ is real. Since $B_\pm^2 = -1$, $Tr(B_\pm) = 0$. Equation (5.7) now follows. $\square$

We shall also need twisted products of odd dimensional orientable manifolds. We refer to Gilkey [20] for the proof of the following results. Clifford multiplication defines an embedding of the spinor groups $Spin(m) \cdot Spin(m) \to Spin(2m)$. Let M be a closed manifold with fundamental group $\mathbb{Z}_\ell$ that admits a $spin^c$ structure whose associated determinant line bundle is given by ρ_b. Let $\tilde{M}$ be the universal cover of M; $\tilde{M}$ admits a natural *spin* structure. Give $\tilde{M} \times \tilde{M}$ the natural *spin* structure. We suppose that the dimension m of M is odd so that the flip $(x, y) \to (y, x)$ reverses the orientation. Since g_ℓ preserves the orientation, $g_{2\ell}$ reverses the orientation so $w_1(N) \neq 0$. Let $\hat{g}_{2\ell}$ be the lift of $g_{2\ell}$ to a morphism of the associated pin^- principal bundle over $\tilde{M} \times \tilde{M}$. Then $\hat{g}_{2\ell}^2 = \pm \hat{g}_\ell \cdot \hat{g}_\ell$ and hence $\hat{g}_{2\ell}^{2\ell} = \hat{g}_\ell^\ell \cdot \hat{g}_\ell^\ell = 1$ so $\tilde{M} \times \tilde{M}$ admits a natural pin^- structure.

5.8 Theorem. *Let M be a closed manifold of dimension $m = 2k+1$ with fundamental group $\mathbb{Z}_\ell$ which admits a spin^c structure s_M with flat associated determinant line bundle given by* $\det(s_M) = \rho_b$. *If $m \equiv 3$ mod 4, let $\beta = 0$; if $m \equiv 1$ mod 4, let $\beta = \ell/2$. Let s_N be the pin$^-$ structure on $N := \tilde{M} \times \tilde{M}/\mathbb{Z}_{2\ell}$ defined above. Let $\delta := (\rho_0 - \rho_{\ell/2})$.*

(a) *If $u = 2v - b + \beta$, then $\eta(N)(\rho_u) = \eta(M)(\rho_v\delta)$ in $\mathbb{R}/\mathbb{Z}$.*

(b) *If $u = 2v - b + \beta + 1$, then $\eta(N)(\rho_u) = 0$ in $\mathbb{R}/\mathbb{Z}$.*

(c) *If there are no harmonic spinors on $\tilde{M}$, the equalities in (a) and in (b) hold in $\mathbb{R}$ not just $\mathbb{R}/\mathbb{Z}$.*

When we apply this result to lens spaces and to lens space bundles, we get:

5.9 Corollary.

(1) *If $m \equiv 3$ mod 4, then $\eta(N(L^m(\ell;\vec{a})), \rho_{2v}) = \eta(L^m(\ell;\vec{a}), \delta\rho_v)$.*

(2) *If $m \equiv 3$ mod 4, then $\eta(N(X^m(\ell;\vec{a})), \rho_{2v-1}) = \eta(X^m(\ell;\vec{a}), \delta\rho_v)$.*

(3) *If $m \equiv 1$ mod 4, then $\eta(N(L^m(\ell;\vec{a})), \rho_{2v-1+\ell/2}) = \eta(L^m(\ell;\vec{a}), \delta\rho_v)$.*

(4) *If $m \equiv 1$ mod 4, then $\eta(N(X^m(\ell;\vec{a})), \rho_{2v+\ell/2}) = \eta(X^m(\ell;\vec{a}), \delta\rho_v)$.*

(5) *Otherwise $\eta(N(L^m(\ell;\vec{a})), \rho_u) = 0$ and $\eta(N(X^m(\ell;\vec{a})), \rho_u) = 0$.*

We apply the previous formulas to the twisted product of real projective spaces:

5.10 Corollary.

(1) $\eta(N(\mathbb{R}P^{4k}))(\rho_{2s+2k}) = \eta(\mathbb{R}P^{4k})(\rho_s) = (-1)^s 2^{-2k-1}$.

(2) $\eta(N(\mathbb{R}P^{4k+1}))(\rho_{2s}) = \eta(\mathbb{R}P^{4k+1})(\rho_s(\rho_0 - \rho_1)) = (-1)^s 2^{-2k-1}$.

(3) $\eta(N(\mathbb{R}P^{4k+2}))(\rho_{2s+2k-1}) = \eta(\mathbb{R}P^{4k+2})(\rho_s) = (-1)^s 2^{-2k-2}$.

(4) $\eta(N(\mathbb{R}P^{4k+3}))(\rho_{2s}) = \eta(\mathbb{R}P^{4k+3})(\rho_s(\rho_0 - \rho_1)) = (-1)^s 2^{-2k-2}$.

Proof. We take $b = 0$ in (1) and $b = 1$ in (3) and apply Remark 5.4 to establish the first equality; the second equality then follows from computations of Gilkey [20]. We take $\beta = 1$ and $b = 1$ in (2) and we take $\beta = 0$ and $b = 0$ in (4) to establish the first equality; the second equality then follows from Theorem 3.2. $\square$

CHAPTER VI

COMPUTING CONNECTIVE K THEORY GROUPS

VI.1 Introduction

In this section, we will compute the additive structure of the connective K theory groups $ko_*(B\mathbb{Z}_4, \xi_0)$ and $ko_*(B\mathbb{Z}_4, \xi_1)$. We will also use the eta invariant to express the connective K theory groups $ko_*(B\mathbb{Z}_\ell) = ko_{4k-1}(B\mathbb{Z}_\ell, \xi_0)$ in terms of the representation theory. We will show that $ko_{4k-1}(B\mathbb{Z}_\ell)$ is isomorphic to the reduced symplectic K theory groups $\widetilde{KSp}(M^{4k+3})$ where $M^{4k+3} := L^{4k+3}(\ell; \vec{a})$ is any lens space of dimension $4k+3$. We will also express the groups $ko_{4k+1}(B\mathbb{Z}_\ell) \cap \ker \hat{\mathbb{A}}$ in terms of representation theory of the group $\mathbb{Z}_\ell$.

VI.2 Orders of the Reduced Connective K Theory Groups

In order to compute $ko_*(B\mathbb{Z}_4, \xi_i)$ for $i = 0, 1$ we will use a computation of the orders of the connective K theory groups by Botvinnik and Gilkey [11]; their calculation, which used the Adams spectral sequence, is crucial to our work. Let $\mathcal{K}(m, \xi_0) := |\widetilde{ko}_m(B\mathbb{Z}_4, \xi_0)|$ and let $\mathcal{K}(m, \xi_i) := |ko_m(B\mathbb{Z}_4, \xi_i)|$ for $i = 1, 2, 3$.

Table 6.1

	$\mathcal{K}(m,\xi_0)$	$\mathcal{K}(m,\xi_1)$	$\mathcal{K}(m,\xi_2)$	$\mathcal{K}(m,\xi_3)$
$m=8k$	1	1	2^{2k+1}	2^{2k+1}
$m=8k+1$	$2(\ell/2)^{2k+1}$	$(2\ell)^{2k+1}$	2	1
$m=8k+2$	2	1	2^{2k+3}	2^{2k+1}
$m=8k+3$	$2(2\ell)^{2k+1}$	$(\ell/2)^{2k+1}$	2	1
$m=8k+4$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+5$	$(\ell/2)^{2k+2}$	$(2\ell)^{2k+2}$	1	1
$m=8k+6$	1	1	2^{2k+2}	2^{2k+2}
$m=8k+7$	$(2\ell)^{2k+2}$	$(\ell/2)^{2k+2}$	1	1

Recall that in §3, we gave the generators for these connective K theory groups, see Lemma 3.3 and Lemma 3.9. Notice that for $i=0$, we have to consider the effect of the $\hat{A}$-genus. Let $\hat{A}(m,\xi)$ denote the range of the $\hat{A}$ genus; the $\hat{A}$ genus vanishes for $m\equiv 5,6,7 \bmod 8$.

Table 6.2

	$\hat{A}(m,\xi_0)$	$\hat{A}(m,\xi_1)$	$\hat{A}(m,\xi_2)$	$\hat{A}(m,\xi_3)$
$m=8k$	$\mathbb{Z}$	–	$\mathbb{Z}_2$	–
$m=8k+1$	$\mathbb{Z}_2$	–	$\mathbb{Z}_2$	–
$m=8k+2$	$\mathbb{Z}_2$	–	$\mathbb{Z}_2$	–
$m=8k+3$	–	–	$\mathbb{Z}_2$	–
$m=8k+4$	$\mathbb{Z}$	–	–	–

In Lemma 2.8, we studied $ko_1(B\mathbb{Z}_\ell,\xi_i)$ for $i=0,1$. We now perform a similar analysis for $ko_2(B\mathbb{Z}_\ell,\xi_i)$ for $i=2,3$.

6.3 Lemma.

(1) $ko_2(B\mathbb{Z}_\ell,\xi_2)=MSpin_2(B\mathbb{Z}_\ell,\xi_2)=\mathbb{Z}_2\oplus\mathbb{Z}_4$.

(2) $ko_2(B\mathbb{Z}_\ell,\xi_2)\cap\ker(\hat{\mathbb{A}})=MSpin_2(B\mathbb{Z}_\ell,\xi_2)\cap\ker(\hat{\mathbb{A}})=\mathbb{Z}_4$.

(3) $ko_2(B\mathbb{Z}_\ell,\xi_3)=MSpin_2(B\mathbb{Z}_\ell,\xi_3)=\mathbb{Z}_2$.

Proof. We adopt the notation of §2.6. Let $M_i^2 := N(S^1, s_i, f_1)$ for $i = 0, 1$. We use equation (2.7), and Theorem 5.8 to see:

$$\eta(M_0^2)(\rho_0) = \eta(S^1, s_0, f_1)(\rho_0 - \rho_1) = -1/2 \text{ in } \mathbb{R}/\mathbb{Z},$$

$$\eta(M_1^2)(\rho_0) = \eta(S^1, s_1, f_1)(\rho_0 - \rho_1) = -1/2 \text{ in } \mathbb{R}/\mathbb{Z},$$

We have $\hat{\mathbb{A}}(M_0^2) = \hat{A}(T^2, s_0) = 0$, and $\hat{\mathbb{A}}(M_1^2) = \hat{A}(T^2, s_1) = 1$. Therefore

$$\eta(M_0^2)(\rho_0) = 1/2,\ \eta(M_0^2 - M_1^2)(\rho_0) = 0 \text{ in } \mathbb{R}/\mathbb{Z}$$

$$\hat{\mathbb{A}}(M_0^2) = 0,\ \hat{\mathbb{A}}(M_0^2 - M_1^2) = 1 \text{ in } \mathbb{Z}_2.$$

By Theorem 2.4, $\eta(\cdot)(\rho_0)$ defines an $\mathbb{R}/2\mathbb{Z}$ valued invariant of $ko_2(B\mathbb{Z}_\ell, \xi_2)$. The first two assertions now follow. The final assertion follows from Table 6.1. □

Let $\vec{a}_{2k} := (1, -1, ..., 1, -1)$. We define manifolds $M_1^{4k+1} := X^{4k+1}(4; \vec{a}_{2k-2}, 1, 1)$ $M_2^{4k+3} := L^{4k+3}(4; \vec{a}_{2k}, 1, 1)$, $M_3^{4k+3} := L^{4k+3}(4; \vec{a}_{2k}, 1, 3)$. These manifolds belong to $ko_m(B(\mathbb{Z}_4))$. We use the result of Donnelly [15], contained in Lemma 3.2, to compute the eta invariant:

Table 6.4

	ρ_0	ρ_1	ρ_2	ρ_3
M_1^{4k+1}	0	-2^{-k-1}	0	2^{-k-1}
M_2^{4k+3}	$-2^{-2k-4} - 2^{-k-2}$	2^{-2k-4}	$-2^{-2k-4} + 2^{-k-2}$	2^{-2k-4}
M_3^{4k+3}	$2^{-2k-4} - 2^{-k-2}$	-2^{-2k-4}	$2^{-2k-4} + 2^{-k-2}$	-2^{-2k-4}

Similarly, define manifolds $N_1^{4k+1} := L^{4k+1}(4; \vec{a}_{2k}, 1)$,$N_2^{4k+1} := L^{4k+1}(4; \vec{a}_{2k}, 3)$, and $N_3^{4k+3} := X^{4k+3}(4; \vec{a}_{2k}, 1)$ in $ko_m(B\mathbb{Z}_4, \xi_1)$. Introduce constants $\alpha_k := 2^{-2k-3}$ and $\beta_k := 2^{-k-2}$. Then we have

Table 6.5

	ρ_0	ρ_1	ρ_2	ρ_3
N_1^{4k+1}	$-\alpha_k - \beta_k$	$\alpha_k - \beta_k$	$-\alpha_k + \beta_k$	$\alpha_k + \beta_k$
N_2^{4k+1}	$\alpha_k - \beta_k$	$-\alpha_k - \beta_k$	$\alpha_k + \beta_k$	$-\alpha_k + \beta_k$
N_3^{4k+3}	$-\beta_k$	β_k	β_k	$-\beta_k$

The $\hat{\mathbb{A}}$ genus plays a special role if $m = 8k+1$ and $\xi = \xi_0$ or if $m = 8k+2$ and if $\xi = \xi_2$. Let

$$to_m(B\mathbb{Z}_4, \xi) := \ker(\hat{A}) \cap ko_m(B\mathbb{Z}_4, \xi).$$

The $\hat{\mathbb{A}}$ genus defines short exact sequences:

$$0 \to to_{8k+1}(B\mathbb{Z}_\ell) \to \widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell) \to \mathbb{Z}_2 \to 0, \tag{6.6}$$

$$0 \to to_{8k+2}(B\mathbb{Z}_\ell, \xi_2) \to ko_{8k+2}(B\mathbb{Z}_\ell, \xi_2) \to \mathbb{Z}_2 \to 0. \tag{6.7}$$

6.8 Lemma.

(1) *The sequence in equation* (6.6) *splits if* $k \geq 1$ *and if* $\ell = 4$.

(2) *The sequence in equation* (6.7) *splits if* $k \geq 0$ *for any* ℓ.

Proof. We remark by Lemma 2.8 that equation (6.6) is *not* split if $8k+1 = 1$. To prove equation (6.6) splits for $k \geq 1$, it suffices to exhibit a element $[M^{8k+1}]$ of $\widetilde{ko}_{8k+1}(B\mathbb{Z}_4)$ of order 2 so that $\hat{\mathbb{A}}(M) = 0 \oplus 1$. In §II.21, we showed that $\eta^*(M \times B^8) = \eta^*(M)$ and $\hat{\mathbb{A}}(M \times B^8) = \hat{\mathbb{A}}(M)$. Thus using the periodicity operator defined by taking product with the Bott manifold, it suffices to construct M^9 of order 2 in $\widetilde{ko}_9(B\mathbb{Z}_4)$ so that $\hat{\mathbb{A}}(M^9) = 0 \oplus 1$.

We adopt the notation of §II.17 and table 6.4. Let $M^9 := (S^1, s_0, f_1) \times B^8 + 2M_1^9$. We use Table 6.4, equation (2.7), and equation (2.11) to see in $\mathbb{R}/\mathbb{Z}$ that

$$\eta(M^9)(\rho_0 - \rho_1) = -1/4 + 2/8 = 0,\ \eta(M^9)(\rho_0 - \rho_2) = 1/2 + 0 = 1/2,$$

$$\eta(M^9)(\rho_0 - \rho_3) = -3/4 - 2/8 = 0, \text{ and } \hat{\mathbb{A}}(M^9) = 0 \oplus 1.$$

By Corollary 3.10, the eta invariant and $\hat{\mathbb{A}}$ completely detect the connective K theory groups, we conclude that M^9 is an element of order 2. This completes the proof of the first assertion.

To prove the second assertion, the same argument shows that it suffices to verify it for $8k+2=2$ i.e. $k=0$. We adopt the notation used to prove Lemma 6.3. We have that $\eta(M_0^2 - M_1^2)(\rho_0) \in \mathbb{Z}$ and $\mathbb{A}(M_0^2 - M_1^2) = 1$ in $\mathbb{Z}_2$. The eta invariant and $\hat{\mathbb{A}}$ genus completely detect $ko_2(B\mathbb{Z}_\ell, \xi_2)$ and the eta invariant is well defined in $\mathbb{R}/2\mathbb{Z}$. Thus $M_0^2 - M_1^2$ has order 2 and has non-vanishing $\hat{\mathbb{A}}$ genus. $\square$

6.9 Remark. Given any k, there exists $\ell(k)$ so that if $\ell > \ell(k)$, then the sequence in (6.6) does not split. We refer to [6] for details; this paper also contains a computation of the connective K theory groups $ko_m(B\mathbb{Z}_\ell)$ for $m = 3, 5, 7, 9$.

The twisted products of real projective spaces are the non-orientable manifolds we will study to compute the twisted connective K theory groups $ko_m(\mathbb{Z}_4, \xi_i)$ for $i = 2, 3$. We take $i = 3$ in dimensions $m \equiv 0 \bmod 4$ since $N(\mathbb{R}P^{2k}) \in ko_{4k}^+(B\mathbb{Z}_4, \xi_3)$. We take $i = 2$ in dimensions $m \equiv 2 \bmod 4$ since $N(\mathbb{R}P^{2k+1}) \in ko_{4k+2}^+(B\mathbb{Z}_4, \xi_2)$.

We can now determine the structure of these connective K theory groups.

6.10 Theorem. *Let $k \geq 1$.*

	$\widetilde{ko}_m(BZ_4)$	$to_m(BZ_4)$	$ko_m(BZ_4, \xi_1)$
$m = 8k+1$	$\mathbb{Z}_{2^{2k+1}} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+3}} \oplus \mathbb{Z}_{2^{2k}}$
$m = 8k+3$	$\mathbb{Z}_{2^{4k+3}} \oplus \mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+3}} \oplus \mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{2k+1}}$
$m = 8k+5$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{4k+5}} \oplus \mathbb{Z}_{2^{2k+1}}$
$m = 8k+7$	$\mathbb{Z}_{2^{4k+5}} \oplus \mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+5}} \oplus \mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{2k+2}}$

Proof. The manifolds M_i and N_i defined above all admit metrics of positive scalar curvature. Thus the discussion in §II.14 shows that the $\hat{A}$-genus of all these manifolds vanishes. Hence these manifolds belong to to_m. We apply Gaussian elimination to Tables 6.4 and 6.5 to determine the range of the eta invariant applied to these manifolds and to obtain a lower bound for subgroups of to_m spanned by these manifolds. We compare this lower bound with the upper bounds contained in table 6.1

to establish the second and third columns given above. The first column differs from the second column only in dimension $m = 8k+1$. The extra factor of $\mathbb{Z}_2$ comes from Table 6.2; to complete the proof, we must show that the extension in question is split. Recall that the extended A-roof genus $\hat{\mathbb{A}}$ genus defined in §2.6 takes values in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Botvinnik, Gilkey, and Stolz [13], the eta invariant and the extended $\hat{\mathbb{A}}$ genus are a complete set of invariants for $ko_{8k+1}(B\mathbb{Z}_4, \xi_0)$. We have

$$\begin{aligned} ko_{8k+1}(B\mathbb{Z}_4, \xi_0) =& ko_{8k+1}(pt) \oplus \widetilde{ko}_{8k+1}(B\mathbb{Z}_4, \xi_0) \\ =& \mathbb{Z}_2 \oplus \widetilde{ko}_{8k+1}(B\mathbb{Z}_4). \end{aligned}$$

The proof of Theorem 6.10 will now follows from Lemma 6.8. □

6.11 Theorem.

	$ko_m(BZ_4, \xi_2)$	$to_m(BZ_4, \xi_2)$	$ko_m(BZ_4, \xi_3)$
$m = 8k$	?	?	$\mathbb{Z}_{2^{2k+1}}$
$m = 8k+2$	$\mathbb{Z}_{2^{2k+2}} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{2^{2k+2}}$	?
$m = 8k+4$	?	?	$\mathbb{Z}_{2^{2k+2}}$
$m = 8k+6$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{2k+2}}$	?

Proof. We use Corollary 5.10 to compute the eta invariants of these manifolds and obtain a lower estimate for the order of the subgroup of to_m generated thereby. We use Table 6.1 to obtain an upper estimate for the orders of these groups. This establishes the result for $to_m(B\mathbb{Z}_4, \xi_2)$ and for $ko_m(B\mathbb{Z}_4, \xi_3)$. Since the short exact sequence (6.79) splits by Lemma 6.8, the result for $ko_m(B\mathbb{Z}_4, \xi_2)$ follows as well. □

The entries '?' are undetermined by this method and are presently under further investigation.

We conclude this section with one of the main results of this thesis. We wish to express the connective K theory groups in terms of the ordinary K theory groups of lens spaces. First we recall some structure theorems which express the unitary and symplectic K theory of lens spaces in terms of the representation theory of the cyclic group. We adopt the notation of §3.1. Let $\tau = \tau(\vec{a})$ be a fixed point free representation of $\mathbb{Z}_\ell$ in $U(k)$ and let $L^{2k-1}(\ell, \vec{a})$ be the associated lens space. If $\rho \in RU_0(\mathbb{Z}_\ell)$, let V_ρ be the associated flat unitary virtual vector bundle over $L^{2k-1}(\ell, \vec{a})$ as defined in §II.8. We refer to [19, §2.5] for the proof of the following result

6.13 Theorem. *Let $M := L^{2k-1}(\ell, \vec{a})$ and let $\rho \in RU_0(\mathbb{Z}_\ell)$.*

(1) *The map $\rho \to [V_\rho]$ induces a surjective ring homomorphism from $RU_0(\mathbb{Z}_\ell)$ to $\widetilde{KU}(M)$.*

(2) *We have $\rho \in RU_0(\mathbb{Z}_\ell)^k \iff \eta(M)(\rho\tilde{\rho}) \in \mathbb{Z}\ \forall\ \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$.*

(3) *We have $[V_\rho] = 0$ in $\widetilde{KU}(M) \iff \eta(M)(\rho\tilde{\rho}) = 0$ in $\mathbb{R}/\mathbb{Z}\ \forall\ \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$.*

(4) *The map $\rho \to [V_\rho]$ defines an isomorphism from $RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^k$ to $\widetilde{KU}(M)$.*

(5) *The map $(\rho, \tilde{\rho}) \to \eta(M)(\rho\tilde{\rho})$ extends to a non-singular symmetric pairing from $\widetilde{KU}(M) \otimes \widetilde{KU}(M)$ to $\mathbb{Q}/\mathbb{Z}$ which exhibits $\widetilde{KU}(M)$ as its own Poincare dual.*

There is a similar characterization of the symplectic K theory groups of a lens space, $\widetilde{KSp}(L^{2k-1}(\ell; \vec{a}))$. Let $c_{Sp} : RSp(\mathbb{Z}_\ell) \to RU(\mathbb{Z}_\ell)$ be the natural injective homomorphism obtained by forgetting the symplectic structure to get a complex structure. Then

$$c_{Sp}(RSp(\mathbb{Z}_\ell)) = span_{\mathbb{Z}}\{(\rho_s + \rho_{-s}) : s \in \mathbb{Z}_\ell^*\}. \tag{6.14}$$

Here $\rho_s(\lambda) = \lambda^s$ defines a linear representation of $\mathbb{Z}_\ell$. Note that although $RSp(\mathbb{Z}_\ell)$ is *not* a ring, $c_{Sp}(RSp(\mathbb{Z}_\ell))$ is a ring. Also note that $c_{Sp}(RSp_0(\mathbb{Z}_\ell))$ is an ideal of $c_{Sp}(RSp(\mathbb{Z}_\ell))$ and that

$$c_{Sp}(RSp_0(\mathbb{Z}_\ell)) = c_{Sp}(RSp(\mathbb{Z}_\ell)) \cap RU_0(\mathbb{Z}_\ell). \tag{6.15}$$

As in Lemma 3.4, we define $\psi := \rho_1 + \rho_{-1} - 2\rho_0 \in c_{Sp}(RSp_0(\mathbb{Z}_\ell))$. We define an extended eta invariant on $RSp_0(\mathbb{Z}_\ell)$ as follows. Let $M = L^{2k-1}(\ell;\vec{a})$ and let $\phi \in RSp_0(\mathbb{Z}_\ell)$.

(1) If $2k-1 \equiv 1 \bmod 4$, let $\check{\eta}(M)(\phi) = \eta(M)(c_{Sp}(\phi))$.

(2) If $2k-1 \equiv 3 \bmod 8$, let $\check{\eta}(M)(\phi) = 0$.

(3) If $2k-1 \equiv 7 \bmod 8$, let $\check{\eta}(M)(\phi) = (\ell/2)\eta(M)(c_{Sp}(\phi))$.

If $\rho \in RSp_0(\mathbb{Z}_\ell)$, let V_ρ be the associated flat symplectic virtual vector bundle over $L^{2k-1}(\ell;\vec{a})$. We refer to [19, §2.7] for the proof of the following result.

6.16 Theorem. *Let $M := L^{2k-1}(\ell;\vec{a})$ and let $\rho \in RSp_0(\mathbb{Z}_\ell)$.*

(1) *The map $\rho \to [V_\rho]$ induces a surjective homomorphism from $RSp_0(\mathbb{Z}_\ell)$ to $\widetilde{KSp}(M)$.*

(2) $[V_\rho] = 0$ *in* $\widetilde{KSp}(M) \iff$

$\eta(M)(c_{Sp}(\rho)\tilde{\rho}) = 0$ *in* $\mathbb{R}/\mathbb{Z}$ $\forall \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$ *and* $\check{\eta}(M)(\rho) = 0$.

(3) *If $2k-1 \equiv 1$ mod 4, then $c_{Sp} : \widetilde{KSp}(M) \to \widetilde{KU}(M)$ is injective;*

$\widetilde{KSp}(M) = RSp_0(\mathbb{Z}_\ell)/c_{Sp}^{-1}RU_0(\mathbb{Z}_\ell)^{k+1}$.

(4) *If $2k-1 \equiv 3$ mod 8, then $c_{Sp} : \widetilde{KSp}(M) \to \widetilde{KU}(M)$ is injective;*

$\widetilde{KSp}(M) = RSp_0(\mathbb{Z}_\ell)/c_{Sp}^{-1}RU_0(\mathbb{Z}_\ell)^{k}$.

(5) *If $2k-1 \equiv 7$ mod 8, then $\ker(c_{Sp}) = \mathbb{Z}_2$;*

$\widetilde{KSp}(M) = RSp_0(\mathbb{Z}_\ell)/c_{Sp}^{-1}\{\psi^{k/2}c_{Sp}RSp_0(\mathbb{Z}_\ell)\}$.

We define:

$$\mathcal{I} := \{\rho \in RU_0(\mathbb{Z}_\ell) : \rho(\bar{\lambda}) = -\rho(\lambda)\}.$$

6.17 Theorem. *Let $m = 2k - 1$.*

(1) *If $m \equiv 1$ mod 8 and $m \neq 1$ then $ko_m(B\mathbb{Z}_\ell) \cap \ker \tilde{\mathbb{A}} \approx \mathcal{I}/\{RU_0(\mathbb{Z}_\ell)^k \cap \mathcal{I}\}$.*

(2) *If $m \equiv 3$ mod 8, then $ko_m(B\mathbb{Z}_\ell) \approx \widetilde{KSp}(L^{m+4}(\ell, \vec{a}))$ for any $\vec{a}$.*

(3) *If $m \equiv 5$ mod 8, then $ko_m(B\mathbb{Z}_\ell) \approx \mathcal{I}/\{RU_0(\mathbb{Z}_\ell)^k \cap \mathcal{I}\}$.*

(4) *If $m \equiv 7$ mod 8, then $ko_m(B\mathbb{Z}_\ell) \approx \widetilde{KSp}(L^{m+4}(\ell, \vec{a}))$ for any $\vec{a}$.*

6.18 Remark. We do not have an expression for $\widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell)$ in terms of the representation theory. We have a short exact sequence

$$0 \to \widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell) \cap \ker \mathbb{A} \to \widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell) \to \mathbb{Z}_2 \to 0.$$

This splits if $\ell = 4$. In joint work with Gilkey [6] we have shown that this sequence does not always split. We do not know if $\widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell)$ is expressible in terms of the representation theory; this problem is under further investigation.

Proof. We adopt the notation of §III.2 and let $\eta^* : ko_{4k-1}(B\mathbb{Z}_\ell) \to RU_0(\mathbb{Z}_\ell)^*$ be defined by $\eta^*([M])(\rho) = \eta(M)(\rho)$. By Theorem 3.9, η^* is injective. We complete the proof of (1) by determining the image of η^*.

Let $M^L_{4k-1,j}$ be as defined in §III.2. By Theorem 3.9, $\widetilde{ko}_{4k-1}(B\mathbb{Z}_\ell)$ is generated by the classes $[M^L_{4k-1,j}]$. We defined $\psi = -(\rho_0 - \rho_1)(\rho_0 - \rho_{-1}) \in RU_0(\mathbb{Z}_\ell)^2$. Let

$$W^{4k+3} := L^{4k+3}(\ell; 3, 3, ..., 3, 3, 1, 1).$$

Then by Lemmas 3.4 and 3.5,

$$\eta(M^L_{4k-1,j})(\rho) = \eta(M^L_{4k-1,0})(\psi^j \rho) = \eta(W^{4k+3})(\psi^{j+1}\rho).$$

We define $\sigma(M^L_{4k-1,j}) := \psi^{j+1}$ and extend σ linearly to the free Abelian group $\mathcal{A}_{4k-1}$ generated by the $M^L_{4k-1,j}$ for $0 \leq j \leq 2k$. We then have

$$\eta(M)(\rho) = \eta(W^{4k+3})(\sigma(M)\rho) \;\forall M \in \mathcal{A}_{4k-1}. \tag{6.19}$$

If $[M]=0$ in $ko_{4k-1}(B\mathbb{Z}_\ell)$, then $\eta(M)(\rho)\in\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$. This implies $\eta(W^{4k+3})(\sigma(M)\rho)\in\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$. By Theorem 6.13, this implies that $\sigma(M)\in RU_0(\mathbb{Z}_\ell)^{2k+2}$. Thus we may regard σ as a well defined map

$$\sigma : ko_{4k-1}(B\mathbb{Z}_\ell)\to RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+2}.$$

6.20 The case $m=4k-1\equiv 7$ mod 8. We use Corollary 3.10 and equation (6.19) to see that the following conditions are equivalent:

(1) $[M]=0$ in $ko_m(B\mathbb{Z}_\ell)$.

(2) $\eta(M)(\rho)=0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$.

(3) $\eta(W^{4k+3})(\sigma(M)\rho)=0\in\mathbb{R}/\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$.

Thus by Theorem 6.13, σ is an injective homomorphism from $ko_{4k-1}(B\mathbb{Z}_\ell)$ to $RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+2}$. We can express $\psi=-(\rho_0-\rho_1)(\rho_0-\rho_{-1})=\rho_1+\rho_{-1}-2\rho_0$. We use equations (6.14) and (6.15) to see that

$$c_{Sp}(RSp_0(\mathbb{Z}_\ell))=\mathrm{Span}_{\mathbb{Z}}\{\psi^{j+1}:j\geq 0\}.$$

Since we are working modulo $RU_0(\mathbb{Z}_\ell)^{2k+2}$, we may restrict to $j+1\leq k$ or equivalently $0\leq j\leq k-1$. Since $\psi^j=\sigma(M^L_{m,j})$ for j in this range, the desired result now follows from Theorem 6.16.

6.21 The case $m=4k-1\equiv 3$ mod 8. We use Corollary 3.10 and equation (6.19) to see that the following conditions are equivalent:

(1) $[M]=0$ in $ko_m(B\mathbb{Z}_\ell)$.

(2) $\eta(M)(\rho)=0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$ and $\eta(M)(\rho_0-\rho_{\ell/2})=0$ in $\mathbb{R}/2\mathbb{Z}$.

(3) $\eta(W^{4k+3})(\sigma(M)\rho)=0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho\in RU_0(\mathbb{Z}_\ell)$ and

$\eta(W^{4k+3})(\sigma(M)(\rho_0-\rho_{\ell/2}))=0$ in $\mathbb{R}/2\mathbb{Z}$.

Let $\rho \in RSp_0(B\mathbb{Z}_\ell)$. By Theorem 6.16, the following conditions are equivalent:

(a) $[V_\rho]$ is 0 in $\widetilde{KSp}(W^{4k+3})$.

(b) $\eta(W^{4k+3})(c_{Sp}(\rho)\tilde{\rho}) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$ and $\ell/2\eta(W^{4k+3})(c_{Sp}\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$.

We must relate conditions (3) and (b) which are given above. Let

(i-a) $\eta(W^{4k+3})(\sigma(M)\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$ and

(i-b) $\eta(W^{4k+3})(\sigma(M)(\rho_0 - \rho_{\ell/2}))/2 = 0$ in $\mathbb{R}/\mathbb{Z}$

(ii-a) $\eta(W^{4k+3})(\sigma(M)\tilde{\rho}) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$ and

(ii-b) $\ell/2\eta(W^{4k+3})(\sigma(M)) = 0$ in $\mathbb{R}/\mathbb{Z}$.

To show that σ extends as an injective map from $ko_{4k-1}(B\mathbb{Z}_\ell)$ to $\widetilde{KSp}(W^{4k+3})$, we must show that conditions (i-a,i-b) and (ii-a,ii-b) are equivalent. Let

$$\rho := ((\ell/2) - 1)\rho_0 - \rho_1 - ... - \rho_{\ell/2-1}.$$

It suffices to prove

$$\eta(W^{4k+3})(\sigma(M)(\rho_0 - \rho_{\ell/2}))/2 \tag{6.22}$$

$$= \ell/2\eta(W^{4k+3})(\sigma(M)) - \eta(W^{4k+3})(\sigma(M)\rho). \tag{6.23}$$

Let $\mathcal{G}(\lambda) := \mathcal{G}_L(3, 3, ..., 3, 3, 1, 1)(\lambda)$. We use the formulas from §3.2 to see that:

$$\eta(W^{4k+3})(\varrho) = \ell^{-1}\Sigma_{\lambda \neq 1}\varrho(\lambda)\mathcal{G}(\lambda) \text{ if } \varrho \in RU(\mathbb{Z}_\ell). \tag{6.24}$$

Let S_1 be the primitive ℓ^{th} roots of unity and let S_2 be the remaining ℓ^{th} roots of unity different from 1. We study the sums defining (6.22) and (6.23). If we sum over S_1, we note that the complex conjugate of $\lambda + \lambda^2 + ... + \lambda^{\ell/2-1}$ is $-\lambda - \lambda^2 - ... - \lambda^{\ell/2-1}$.

The crucial point is that $\mathcal{G}(\lambda)$, $\sigma(M)(\lambda)$, and $\rho_{\ell/2}(\lambda)$ are *real.* Thus the terms involving $\rho_1 + ... + \rho_{\ell/2-1}$ for λ cancels the terms for $\bar{\lambda}$ and play no role when we sum over S_1. Furthermore $(\rho_0 - \rho_{\ell/2})(\lambda) = 2$ if $\lambda \in S_1$. Thus

$$\begin{aligned}
&(6.25) && \ell^{-1}\Sigma_{\lambda\in S_1}\mathcal{G}(\lambda)\sigma(M)(\lambda)(\rho_0 - \rho_{\ell/2})(\lambda)/2 \\
& && =\ell^{-1}\Sigma_{\lambda\in S_1}\mathcal{G}(\lambda)\sigma(M)(\lambda) \\
&(6.26) && =\ell^{-1}\Sigma_{\lambda\in S_1}\mathcal{G}(\lambda)\sigma(M)(\ell/2\rho_0 - \rho)(\lambda).
\end{aligned}$$

We have $S_2 = \mathbb{Z}_\ell - S_1 - \{1\} = \mathbb{Z}_{\ell/2} - \{1\}$. If $\lambda \in \mathbb{Z}_{\ell/2}$ and $\lambda \neq 0$, then

$$\rho(\lambda) = \ell/2 - (\rho_0 + ... + \rho_{\ell/2-1})(\lambda) = \ell/2$$

$\rho_0 + ... + \rho_{\ell/2-1}$ is the regular representation of $\mathbb{Z}_{\ell/2}$. Therefore

$$\begin{aligned}
&(6.27) && \ell^{-1}\Sigma_{\lambda\in S_2}\mathcal{G}(\lambda)\sigma(M)(\lambda)(\rho_0(\lambda) - \rho_{\ell/2}(\lambda))/2 \\
& && =0 \\
&(6.28) && =\ell^{-1}\Sigma_{\lambda\in S_2}\mathcal{G}(\lambda)(\ell/2 - \rho(\lambda)).
\end{aligned}$$

We use equation (6.24), the equality of (6.22) with (6.23) and of (6.25) with (6.26) to establish the equality of (6.22) with (6.23).

To show that σ is surjective, note that if $c_{Sp}(\rho) \in RU_0(\mathbb{Z}_\ell)^{2k+4}$, then $[V_\rho] = 0$ in $\widetilde{KSp}(W^{4k+3})$. Thus we can work in $RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+4}$ and that permits us to restrict to $1 \leq j+1 \leq k+1$ i.e. $0 \leq j \leq k$.

6.29 The case $m = 4k+1 \equiv 5$ **mod** 8**.** Let $M^X_{4k+1,j}$ be as defined in §III.2. By Theorem 3.6, $\widetilde{ko}_{4k+1}(B\mathbb{Z}_\ell)$ is generated by the classes $[M^X_{4k+1,j}]$. As in Lemma 3.4, define $\psi = -(\rho_0 - \rho_1)(\rho_0 - \rho_{-1}) \in RU_0(\mathbb{Z}_\ell)^2$. Let

$$Y^{4k+3} := L^{4k+3}(\ell; 1, 3, ..., 3, 3, 3, 1, -1).$$

Then by Lemmas 3.4 and 3.5,

$$\eta(M^X_{4k+1,j})(\rho) = \eta(M^X_{4k+1,0})(\psi^j\rho) = \eta(Y^{4k+3})(\psi^j(\rho_0+\rho_1)(\rho_0-\rho_{-1})\rho).$$

We define $\sigma(M^X_{4k+1,j}) := \psi^j(\rho_0+\rho_1)(\rho_0-\rho_{-1})$ and extend σ linearly to the free Abelian group $\mathcal{A}_{4k+1}$ generated by the $M^X_{4k+1,j}$ for $0 \le j \le 2k-1$. We then have

$$\eta(M)(\rho) = \eta(Y^{4k+3})(\sigma(M)\rho)\ \forall M \in \mathcal{A}_{4k+1}. \tag{6.30}$$

If $[M] = 0$ in $ko_{4k+1}(B\mathbb{Z}_\ell)$, then $\eta(M)(\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$. This implies $\eta(Y^{4k+3})(\sigma(M)\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$. By Theorem 6.13, this implies that $\sigma(M) \in RU_0(\mathbb{Z}_\ell)^{2k+2}$. Thus we may regard σ as a well defined map

$$\sigma : ko_{4k+1}(B\mathbb{Z}_\ell) \to RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+2}.$$

It is immediate from the construction that σ is injective. We complete the proof of (3) by observing that

$$\sigma(ko_{4k+1}(B\mathbb{Z}_\ell)) + RU_0(\mathbb{Z}_\ell)^{2k} = \mathcal{I} + RU_0(\mathbb{Z}_\ell)^{2k}.$$

The proof of (1) is similar since the $\{M^X_{m,j}\}$ generate $ko_m(B\mathbb{Z}_\ell) \cap \ker \tilde{\mathbb{A}}$. $\square$

CHAPTER VII

MODULI SPACES

VII.1 Introduction

The simplest local invariant of a Riemannian metric g_M on a closed manifold M is the scalar curvature $\tau := R_{ijji}$. It is possible to give a very elegant geometric characterization of τ. Let $B_r(x, M))$ be the geodesic ball of radius r about a point $x \in M$. Then

$$vol(B_r(x, M)) = vol(B_r(0, \mathbb{R}^m))(1 - \tau(x)r^2/6m + O(r^4);$$

see [7, (0.60)] for details. Thus if the metric has positive scalar curvature (psc), the volumes of small geodesic balls in M grow less rapidly than they do in flat space $\mathbb{R}^m$.

The following result follows from work of Gromov and Lawson [21] and of Schoen and Yau [34]; it is the fundamental result in differential geometry used to study metrics of positive scalar curvature.

7.1 Theorem. *Let M be a Riemannian manifold which admits a metric of positive scalar curvature. If N can be obtained from M by surgeries in codimension at least 3, then N admits a metric of positive scalar curvature.*

It is beyond the scope of this thesis to recap in full the proof of this theorem. It is, however, worth giving some of the flavor involved. Let S^k be an embedded k

dimensional sphere in M with trivial normal bundle ν. This means that a tubular neighborhood of S^k has the form $S^k \times D^{m-k}$ and associated boundary $S^k \times S^{m-k-1}$. Shrink the size of the tubular neighborhood. It is possible to deform the original metric on M to a metric which is "" in a neighborhood the boundary $S^k \times S^{m-k-1}$ in such a way that the new metric still has positive scalar curvature. It is at this point that the assumption that $m - k \geq 3$ is crucial to ensure that the standard metric on the fiber spheres S^{m-k-1} has positive scalar curvature and this dominates as the size of these spheres is shrunk by taking an adiabatic limit. The surgery can then be performed; one cuts out the $S^k \times \mathrm{int} D^{m-k}$ and glues in a $D^{k+1} \times S^{m-k-1}$ and preserves the positity of the scalar curvature.

VII.2 Moduli Space

In this section, we discuss the moduli space of metrics of positive scalar curvature on a Riemannian manifold of dimension $m \geq 5$ in the spin context. We say that two metrics of positive scalar curvature g_i on M are *concordant* if there exists a metric g on $M \times [0,1]$ which has positive scalar curvature, which is product near the boundary, and which restricts to the given metrics at $M \times i$ for $i = 0, 1$. Let $\mathcal{R}(M)$ be the space of metrics of positive scalar curvature on M and let $\mathcal{M}(M) := \mathcal{R}(M)/\mathrm{Diff}(M)$ be the associated moduli space. Two metrics which are in the same arc component of $\mathcal{R}(M)$ are necessarily concordant; it is not known if the converse holds.

One can apply techniques of algebraic topology to deduce the following Theorem from Theorem 7.1. We refer to Rosenberg [30, 31, 32], Rosenberg and Stolz [33], and to Miyazaki [29] for details; see also Botvinnik and Gilkey [8, 10].

7.2 Theorem. *Let π be a finite group. Let ρ be a virtual representation of π and let ξ be a real vector bundle over the classifying space $B\pi$. If m is even, assume that ξ is non-orientable and that ξ admits a pin^c structure. If m is odd, assume* *that ξ admits a spin^c structure and that ρ has virtual dimension 0. Let M be a connected closed manifold of dimension $m \geq 5$ with $\pi_1(M) = \pi$. Let f be the canonical π structure on M. Assume there exists a spin structure s on $T(M) \oplus f^*\xi$ so $[(M, f, s)] \in MSpin_m(B\pi, \xi)$. Suppose there exists a closed manifold M_1 which admits a metric g_1 of positive scalar curvature so that $[(M, f, s)] = [(M_1, f_1, s_1)]$ in $MSpin_m(B\pi, \xi)$; M_1 need not be connected. Then M admits a metric of positive scalar curvature g so that $[(M, s, f, g)] = [(M_1, s_1, f_1, g_1)]$ in ${}^+MSpin_m(B\pi, \xi)$.*

The following is the basic tool we shall use. It uses Theorem 7.2. As for the proof of Theorem 7.2, we refer to Botvinnik and Gilkey [8, 10]; those authors dealt with the orientable case; the extension to the non orientable setting is immediate and is therefore omitted.

7.3 Theorem. *Let π be a finite group. Let ρ be a virtual representation of π and let ξ be a real vector bundle over the classifying space $B\pi$. If m is even, assume that ξ is non-orientable and that ξ admits a pin^c structure. If m is odd, assume that ξ admits a spin^c structure and that ρ has virtual dimension 0. Let M be a connected closed manifold of dimension $m \geq 5$ with $\pi_1(M) = \pi$. Let f be the canonical π structure on M. Assume there exists a spin structure s on $T(M) \oplus f^*\xi$ so $[(M, f, s)] \in MSpin_m(B\pi, \xi)$.*

(1) *Let $[(M_2, f_2, s_2, g_2)] = 0$ in ${}^+MSpin_m(B\pi, \xi)$. Then $\eta(M_2, \rho) = 0$ in $\mathbb{R}$.*

(2) *Suppose that there exists $[(M_3, f_3, s_3, g_3)]$ in ${}^+MSpin_m(B\pi, \xi)$ such that $\eta(M_3, f_3, s_3, g_3, \rho) \neq 0$ in $\mathbb{R}$. Suppose that M admits a metric of positive scalar curvature. Then $\mathcal{M}(M)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on M which are not geometrically bordant and which are not concordant.*

To apply Theorem 7.3 to study the moduli space of metrics of positive scalar curvature on a Riemannian manifold, we must construct manifolds which admit metrics of positive scalar curvature and which have non-vanishing eta invariant. If σ is a group homomorphism from G to H, we have natural maps

$$\sigma_B : BG \to BH, \ \sigma_R : RH \to RG, \text{ and}$$

$$\sigma_M : MSpin_m(BG, \sigma_B^*\xi) \to MSpin_m(BH, \xi).$$

(When discussing the case m is even, we shall need to assume that both ξ and $\sigma_B^*\xi$ are non-orientable). Inequivalent $spin^c$ structures on ξ_H are parametrized by complex line bundles; there exists a suitable linear representation ρ^ε which reflects choice of the determinant line bundle on $\sigma_B^*\xi$ so that:

$$\eta(\sigma_M(M), \rho) = \eta(M, \rho^\varepsilon \sigma_R(\rho)). \tag{7.4}$$

For example, let σ be the natural surjective map from $\mathbb{Z}_{2\ell}$ to $\mathbb{Z}_\ell$. Then $\sigma_B^*\xi_1 = \xi_0$ and we take $\rho^\varepsilon = \rho_1$. We can use equation (7.4) to reduce the existence of non-trivial eta invariant to a corresponding question concerning cyclic groups in many instances.

The following Theorem follows from work of Botvinnik and Gilkey [8, 10]. It is not necessary to assume the fundamental group is 2 primary.

7.5 Theorem. *Let M be an orientable manifold of odd dimension $m \geq 5$ with non-trivial cyclic fundamental group $\mathbb{Z}_n$ whose universal cover is spin and which admits a metric of positive scalar curvature. If $m \equiv 3$ mod 4 and if $w_2(M) \neq 0$ or if $m \equiv 1$ mod 4 and if $w_2(M) = 0$, assume $n \geq 3$. Then $\mathcal{M}(M)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on M which are not geometrically bordant and which are not concordant.*

The following theorem deals with non-orientable manifolds. Again, we consider fundamental groups $\mathbb{Z}_n$ which need not be 2 primary; if M is not orientable, n is necessarily even. The case $n \equiv 2$ mod 4 is slightly exceptional.

7.6 Theorem. *Let M be a non orientable manifold of even dimension $m \geq 6$ with cyclic fundamental group $\mathbb{Z}_n$ whose universal cover is spin and which admits a metric of positive scalar curvature. Let $n = a \cdot \bar{n}$ where n is odd and where a is 2 primary. Suppose one of the following cases holds:*

1. *$a = 2$, $m \equiv 0$ mod 4, and $w_2 \neq 0$.*
2. *$a = 2$, $m \equiv 2$ mod 4, and $w_2 = 0$.*
3. *$a = 4$, $m \equiv 0$ mod 4, and $w_2 \neq 0$.*
4. *$a \geq 4$, $m \equiv 2$ mod 4, and $w_2 = 0$.*
5. *$a \geq 8$, $m \equiv 4$ mod 8, $m \geq 12$, and $w_2 = 0$.*

Then $\mathcal{M}(M)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on M which are not geometrically bordant and which are not concordant.

Proof. We will apply Theorem 7.3; we must construct a suitable manifold with a non-vanishing eta invariant. We may assume without loss of generality that $n = a$ is 2 primary. We will consider the following manifolds

1. Let $m = 4k$, $\xi = \xi_3$, and $M := RP^{4k}$.
2. Let $m = 4k + 2$, $\xi := \xi_2$, and $M := RP^{4k+2}$.
3. Let $m = 4k$, $\xi = \xi_3$, and $M := (S^{2k} \times S^{2k})/\mathbb{Z}_4$.
4. Let $m = 4k + 2$, $\xi := \xi_2$, and $M := (S^{2k+1} \times S^{2k+1})/\mathbb{Z}_a$.
5. Let $m = 4k + 4$, $\xi = \xi_2$, and $M := ((S^{2k+1} \times S^{2k+1}) \times (S^{2k+1} \times S^{2k+1}))/\mathbb{Z}_a$.

Note that $[M] \in {}^+MSpin_m(B\mathbb{Z}_a, \xi)$. We complete the proof by showing the eta is non-trivial on these examples.

We first consider cases (1) and (2) where M is real projective space. Gilkey [17, Theorem 3.3] gave a direct computation to show that $\eta(\mathbb{R}P^{2j})(\rho_0) = \pm 2^{-j-1}$; there is a small sign ambiguity that depends upon the exact *pin* or *pin*^{c} structure which is chosen and which plays no role here. This result also follows from the fixed point formulas of Donnelly [15] when extended suitably to the non-orientable setting.

Next consider case (3) where M is the twisted product of $\mathbb{R}P^{2k}$ with $\mathbb{R}P^{2k}$. This admits a suitable twisted pin^c structure as was discussed in Example 4.1. We use Remark 5.4 to express the eta invariant of M in terms of the eta invariant of $\mathbb{R}P^{2k}$ computed above and to see that this is non-zero. If k is even, let $b = 0$; if k is odd, let $b = 1$. Then

$$\eta((S^{2k} \times S^{2k})/\mathbb{Z}_4)(\rho_{-b+k}) = \eta(\mathbb{R}P^{2k})(\rho_0) \neq 0.$$

Next consider case (4) where M is the twisted product of two odd dimensional lens spaces. Let $a = 4\bar{a}$. Let b and β be as defined in Theorem 5.8. We compute

$$\eta((S^{2k+1} \times S^{2k+1})/\mathbb{Z}_{4\bar{a}})(\rho_{2v-b+\beta}) = \eta(S^{2k+1}/\mathbb{Z}_{2\bar{a}})(\rho_v(\rho_0 - \rho_{\bar{a}})).$$

If this vanishes identically for all u, then $\eta(S^{2k+1}/\mathbb{Z}_{2\bar{a}})(\rho(\rho_0 - \rho_{\bar{a}})) = 0$ for all ρ. Take $\rho = \mathcal{F}_L$. Then

$$0 = (2\bar{a})^{-1}\tilde{\Sigma}_\lambda(1 - \lambda^{\bar{a}}) = (2\bar{a})^{-1}\Sigma_\lambda(1 - \lambda^{\bar{a}}) = 1/2$$

which is false. Therefore the eta invariant is non-trivial in this case.

Finally consider case (5) where M is a 4 fold twisted product. Let $m = 8k + 4$. Suppose first $a = 8$. We apply Remark 5.4 and Corollary 5.10 to see that

$$\begin{aligned}
&\eta(((S^{2k+1} \times S^{2k+1}) \times (S^{2k+1} \times S^{2k+1}))/\mathbb{Z}_8)(\rho_{-b+4k+4}) \\
=&\eta((S^{2k+1} \times S^{2k+1})/\mathbb{Z}_4)(\rho_0 + \rho_1) = \eta((S^{2k+1} \times S^{2k+1})/\mathbb{Z}_4)(\rho_0) \\
=&\eta(S^{2k+1}/\mathbb{Z}_2)(\rho_0 - \rho_1) = \pm 2^{-k-1} \neq 0.
\end{aligned}$$

Finally suppose that $a = 16\bar{a} \geq 16$. Let b be as defined in Remark 5.4. Then

$$\begin{aligned}
&\eta(((S^{2k+1} \times S^{2k+1}) \times (S^{2k+1} \times S^{2k+1}))/\mathbb{Z}_a)(\rho_{2u-b+m/2+2\bar{a}}) \\
=&\eta((S^{2k+1} \times S^{2k+1})/\mathbb{Z}_{8\bar{a}})(\rho_u + \rho_{u+2\bar{a}}).
\end{aligned}$$

We now take $u = 2v - b + \beta$ where b and β are as defined in Theorem 5.8 to compute

$$\begin{aligned}&\eta((S^{2k+1} \times S^{2k+1})/\mathbb{Z}_{8\bar{a}})(\rho_u + \rho_{u+2\bar{a}})\\ =&\eta(S^{2k+1}/\mathbb{Z}_{4\bar{a}})((\rho_v + \rho_{v+\bar{a}})(\rho_0 - \rho_{2\bar{a}})).\end{aligned}$$

If this vanishes for all v, then $\eta(S^{2k+1}/\mathbb{Z}_{4\bar{a}})(\rho(\rho_0 + \rho_{\bar{a}})(\rho_0 - \rho_{2\bar{a}})) = 0$ for all ρ. Again, taking $\rho = \mathcal{F}_L$ as appropriate, we see

$$0 = (4\bar{a})^{-1}\tilde{\Sigma}_\lambda(1 + \lambda^{\bar{a}})(1 - \lambda^{2\bar{a}}) = (4\bar{a})^{-1}\Sigma_\lambda(1 + \lambda^{\bar{a}})(1 - \lambda^{\bar{a}}) = 1/2$$

which is false. Therefore the eta invariant is non-trivial in this case as well. □

7.7 Remark. This theorem also follows from work of Gilkey [20] who used entirely different methods.

VII.3 Embeddings

Let $\ell \geq 4$. Botvinnik and Gilkey [11] and Botvinnik, Gilkey, and Stolz [13] have shown that the eta invariant and the $\hat{A}$ genus completely detect $ko_m(B\mathbb{Z}_\ell, \xi_i)$ if $i = 0, 1$ and if m is odd; Gilkey [20] has shown the eta invariant and the $\hat{A}$ genus completely detect $ko_m(B\mathbb{Z}_\ell, \xi_2)$ if $m \equiv 2 \bmod 4$. Let $r : \mathbb{Z}_{2\ell} \to \mathbb{Z}_\ell$ be reduction mod ℓ; $r(\lambda) = \lambda^2$. for $\lambda \in \mathbb{Z}_{2\ell}$. This induces a natural map

$$f_r : B\mathbb{Z}_{2\ell} \to B\mathbb{Z}_\ell;$$

$f_r^*(\rho_s) = \rho_{2s}$. This gives rise to maps in bordism and connective K-theory

$$\begin{aligned}&(f_r)_* : MSpin(B\mathbb{Z}_{2\ell}) \to MSpin(B\mathbb{Z}_\ell),\\ &(f_r)_* : ko_m(B\mathbb{Z}_{2\ell}) \to ko(B\mathbb{Z}_\ell)\end{aligned}$$

as we discussed above. Since $f_r^*\xi_1 = \xi_0$, we also have maps

$$\begin{aligned}&(f_r^1)_* : MSpin(B\mathbb{Z}_{2\ell}) \to MSpin(B\mathbb{Z}_\ell),\\ &(f_r^1)_* : ko_m(B\mathbb{Z}_{2\ell}) \to ko(B\mathbb{Z}_\ell).\end{aligned}$$

Geometrically, if M is a *spin* manifold that admits a $\mathbb{Z}_{2\ell}$ structure, we give M the $spin^c$ structure by twisting the *spin* structure by $\rho_1(M)$; the associated determinant line bundle is then $\rho_2(M)$. It is immediate from this discussion that

$$\eta((f_r)_*M)(\rho_s) = \eta(M)(\rho_{2s}), \text{ and}$$
$$\eta((f_r^1)_*M)(\rho_s) = \eta(M)(\rho_{2s+1}).$$

Let L_ℓ be defined by $\rho_{\ell/2}$; since $f_r^*(L_\ell) = L_{2\ell}$,

$$\hat{A}((f_r)_*M) = \hat{A}(M).$$

A similar discussion can be given in the non-orientable setting. We note

$$R(\mathbb{Z}_{2\ell}) = f_r^* R(\mathbb{Z}_\ell) \oplus \rho_1 f_r^* R(\mathbb{Z}_\ell).$$

This then leads to the observation

VII.3 Theorem.

a) *If $m \equiv 1$ mod 4, then* $0 \to ko_m(B\mathbb{Z}_{2\ell}) \to ko_m(B\mathbb{Z}_\ell) \oplus ko_m(B\mathbb{Z}_\ell, \xi_1)$.

b) *If $m \equiv 2$ mod 4, then* $0 \to ko_m(B\mathbb{Z}_{2\ell}, \xi_2) \to ko_m(B\mathbb{Z}_\ell, \xi_2) \oplus ko_m(B\mathbb{Z}_\ell, \xi_3)$.

If $m \equiv 3$ mod 4, we must deal with the augmentation ideal and this argument fails;

$$R_0(\mathbb{Z}_\ell) \neq f_r^* R_0(\mathbb{Z}_\ell) + \rho_1 f_r^* R_0(\mathbb{Z}_\ell).$$

CHAPTER VIII

THE GROMOV LAWSON CONJECTURE

VIII.1 Introduction

If $m = 2$, the Gauss-Bonnet formula relates the Euler-Poincare characteristic $\chi(M)$ to the scalar curvature;

$$\chi(M) = (4\pi)^{-1} \int_M \tau(x) dvol(x).$$

Thus if a 2 dimensional manifold M admits a metric of positive scalar curvature, then $\chi(M) > 0$. This implies that $M = S^2$ or $M = \mathbb{R}P^2$ and of course, these manifolds do admit metrics of positive scalar curvature. Thus the $\chi(M) > 0$ if and only if M admits a metric of positive scalar curvature.

The situation is very different in higher dimensions. In dimensions $m = 3$ and $m = 4$, Seifert-Whitten theory has been used to study the existence question of metrics of positive scalar curvature and we refer to [40] for further details. We shall concentrate henceforth on the case $m \geq 5$.

Recall that if a spin manifold M admits a metric of positive scalar curvature, then there are no harmonic spinors on M. Thus in particular, the $\hat{A}$-roof genus of A vanishes, see §2.6. The Gromov-Lawson conjecture as extended by Rosenberg for a group π proposes that if M is a closed connected manifold of dimension

$m \geq 5$ with fundamental group π whose universal cover $\tilde{M}$ is spin, then M admits a metric of positive scalar curvature if and only if a generalized index of the Dirac operator $\alpha_\pi(M)$ vanishes. In the cases that we will consider, the invariant α_π can be formulated in terms of the $\hat{A}$-genus.

Stolz [37] has proved the original Gromov-Lawson conjecture in the simply connected case. He has proved:

8.1 Theorem. *Let M be a closed simply connected spin manifold of dimension $m \geq 5$. Then M admits a metric of positive scalar if and only if $\hat{A}(M) = 0$.*

Let $MSpin_m^+(B\pi, \xi)$ be the image of ${}^+MSpin_m(B\pi, \xi)$ in the equivariant twisted bordism group $MSpin_m(B\pi, \xi)$; these are the classes which can be represented by manifolds which admit metrics of positive scalar curvature. The invariant α_π extends to the bordism groups $MSpin_m(B\pi, \xi)$; the formula of Lichnerowicz [28] show that it vanishes on $MSpin_m^+(B\pi, \xi)$. One therefore has the following equivalent formulation of the Gromov-Lawson-Rosenberg conjecture, see [11] for details:

8.2 Theorem. *If π is a finite group, if $m \geq 5$, and if ξ is a real vector bundle over $B\pi$, then the following assertions are equivalent:*

(1) *Let M be any closed connected manifold of dimension m with fundamental group π and canonical $B\pi$ structure f os that $T(M) \oplus f^*\xi$ admits a spin structure. Then M admits a metric of positive scalar curvature if and only if $\alpha_\pi(M) = 0$.*

(2) $MSpin_m^+(B\pi, \xi) = \ker(\alpha_\pi) \cap MSpin_m(B\pi, \xi)$.

By a theorem of Kwasik and Schultz [25], the Gromov-Lawson-Rosenberg conjecture holds for a finite group π if and only if the conjecture holds for all the Sylow subgroups of π. Hence it suffices to assume that π is a p group. Theorems 7.1 and

7.3 reduce the question of constructing a metric of positive scalar curvature on M to a question in bordism. Since all the torsion in the spin bordism ring $MSpin_*$ is 2 torsion, the prime 2 plays distinguished role.

Suppose that π is an Abelian 2 group. By Theorem 2.3, we have

$$ko_m(B\pi, \xi) = MSpin_m(B\pi, \xi)/T_m(B\pi, \xi)$$

where $T_m(B\pi, \xi)$ is the subgroup of $MSpin_m(B\pi, \xi)$ generated by $\mathbb{H}P^2$ geometrical fibrations. We use Besse [7, §9.59] to see the total space of such a fibration admits a metric of positive scalar curvature. Thus α_π vanishes on $T_m(B\pi, \xi)$ and extends to the connective K theory groups $ko_m(B\pi, \xi)$. Let $ko_m^+(B\pi, \xi)$ be the image of $MSpin_m^+(B\pi, \xi)$ in $ko_m(B\pi, \xi)$. The Gromov-Lawson-Rosenberg conjecture has the following reformulation in terms of connective K theory:

8.3 Theorem. *If π is an Abelian 2 group, if $m \geq 5$, and if ξ is a real vector bundle over $B\pi$, then the following assertions are equivalent:*

(1) *Let M be any closed connected manifold of dimension m with fundamental group π and canonical $B\pi$ structure f os that $T(M) \oplus f^*\xi$ admits a spin structure. Then M admits a metric of positive scalar curvature if and only if $\alpha_\pi(M) = 0$.*

(2) $ko_m^+(B\pi, \xi) = \ker(\alpha_\pi) \cap ko_m(B\pi, \xi)$.

We can now prove the Gromov-Lawson conjecture for $\pi = \mathbb{Z}_4$

8.4 Theorem. *Let M be a connected closed non-orientable manifold of dimension m with $\pi_1(M) = \mathbb{Z}_4$. Assume that M admits a flat pin^c structure.*

(1) *If $m = 4k \geq 8$ and if $w_2(M) \neq 0$, then M admits a metric of positive scalar curvature.*

(2) *If $m = 4k + 2 \geq 6$ and if $w_2(M) = 0$, then M admits a metric of positive scalar curvature.*

Proof. Recall that by definition we have

$$\begin{aligned} to_{8k+2}(B\mathbb{Z}_4, \xi_2) :=& ko_{8k+2}(B\mathbb{Z}_4, \xi_2) \cap \ker(\hat{A}) \\ =& ko_{8k+2}(B\mathbb{Z}_4, \xi_2) \cap \ker(\alpha_\pi) \end{aligned}$$

By Theorem 8.3, it suffices to show ${}^{+}ko_{8k+2}(B\mathbb{Z}_4, \xi_2) = to_{8k+2}(B\mathbb{Z}_4, \xi_2)$. This follows from Theorem 6.11. This proves the second assertion; the first follows similarly. $\square$

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