Analysis Qualifying Exam (W2011)

1. Let f_n and f be functions on $L^1([0,1])$. Suppose that $f_n \to f$ a.e.m. Prove that, for any $\epsilon > 0$ there is a compact subset $F \subset [0,1]$ with $m(F) > 1-\epsilon$ such that

$$\lim_{n\to\infty} \int_F |f_n - f| dm = 0.$$

2. Let $f \in L^1([0,1])$. Find

$$\lim_{n\to\infty} \int_{[0,1]} \frac{(x/n)^{5/2}}{1-\cos(x/n)} f(x) dm(x).$$

(Justify your answer!)

3. Let $f \in L^1(\mu)$. Prove that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_E |f| d\mu < \epsilon$$

for any measurable set E with $\mu(E) < \delta$.

4. Let X be a compact metric space and let $\{f_n\} \subset C(X)$ be a sequence of continuous functions. Suppose that

$$\lim_{n\to\infty} F(f_n) = 0$$

for every bounded linear functional F of $C(X)$. Prove that, for any $\sigma > 0$,

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n| \geq \sigma\}) = 0$$

for every Borel measure μ.

5. Let $(X, \mathcal{R}, \mu)$ be a measure space and let f be a complex measurable function. Suppose that, for any $g \in L^1(\mu)$, $fg \in L^1(\mu)$. Prove that there is a constant $M > 0$ such that $|f(x)| \leq M$ *a.e.*μ.

6. Let X be a Banach space, let $X_0 \subset X$ be a closed subspace and let $x_0 \in X_0$ be a non-zero vector. Suppose that $T : X_0 \to X_0$ is a bounded operator such that the range of T is contained in the subspace spanned by x_0. Prove that there exists a bounded operator $T_1 : X \to X$ such that $T_1|_{X_0} = T$.

7. Suppose that f is holomorphic on $\mathbb{C} \setminus \{0\}$ and there exists an integer $N \geq 1$ such that

$$|f(z)| \leq C|z|^{-N} \qquad \text{for all } z \neq 0.$$

Prove that f is a rational functional with a pole at 0 of order at most N.

8. The disc algebra $A(D)$, where D is the open unit disk, consists of continuous functions in $\bar{D}$ and holomorphic in D with the usual sup norm

$$||f|| = \sup_{z \in \bar{D}} |f(z)|.$$

Prove that $A(D)$ is a closed subspace of $C(\bar{D})$.

9. Let $n \geq 2$ be an integer and $f(z) = z^n + 2z^{n-1} + 3z^{n-2} + \cdots + nz + n + 1$. Let A be the set of zeros of f. Find

$$\sum_{a_j \in A} Res(\frac{1}{f}, a_j).$$

Justify your answer.