Winter

Analysis Qualifying Examination

~~S 2009~~ W 2010

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let X be a compact metric space. Suppose that μ is a regular Borel measure on X and suppose that f is a Borel function such that

$$\int_F f d\mu = 0$$

for all closed subsets F of X. Show that $f = 0$ a.e. μ.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of measurable functions which converges to zero almost everywhere (with respect to μ). Prove that f_n converges to zero in measure.

3. Let $\{f_n\}$ be a sequence of Lebesgue integrable functions on the real line. Suppose that $|f_n(x)| \leq 1$ for all $x \in \mathbf{R}$ and f_n converges to zero almost everywhere. Is it true that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n dm = 0,$$

where m is the Lebesgue measure? Why?

4. Let f be a non-decreasing function on $[a, b]$. Prove that f' is in $L^1([a, b])$ and

$$\int_{[a,b]} f'(x)dx \leq f(b) - f(a).$$

5. Let P_k be the set of polynomials of degree at most k on $\mathbf{C}$. Suppose that $|z_1|, |z_2| \leq 1$. Prove that there exists a Borel measure μ on the unit disk

such that

$$\int_{\bar{D}} p(z) d\mu(x) = p'(z_1) + p(z_2)$$

for all $p \in P_k$.

6. Prove that no infinite dimensional Hilbert space has countable algebraic basis.

7. Let $C(\bar{D})$ be the Banach space of all complex valued continuous functions on $\bar{D}$. . Let $H \subset C(\bar{D})$ be the subset consisting of functions which are holomorphic in the open unit disk. Show that H is not dense in $C(\bar{D})$.

8. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6} dx.$$

9. Show that if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then there must be at least one point z with $|z| = 1$ for which $|p(z)| \geq 1$.