Qualifying Exam Analysis Winter 2008

Name: **ID Number:**

There are 9 questions in this exam.

Problem 1. Let $E \subset [0,1]$ be a Lebesque measurable set with $m(E) > 0$. Show that there is a subset $F \subset E$ which is not Lebesque measurable.

Problem 2. Let $\{f_n\}$ be a sequence in $L^1(X,\mu)$ satisfying

$$\lim_{n\to\infty} \int_S |f_n| d\mu = 0$$

for all measurable subsets $S \subset X$. Assume there is a $M > 0$ such that $|f_n(x)| \leq M$ for all n and $x \in X$. Show that, for any $f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \int_X |f f_n| d\mu = 0.$$

Problem 3. Show the following equalities ($a > 0$).
(i) $\int_0^\infty e^{-nx} \sin(ax) dx = \frac{a}{n^2+a^2}$ for any $n > 0$.

(ii) $\int_0^\infty \frac{\sin(ax)}{e^x - 1} dx = \sum_{n=1}^\infty \frac{a}{n^2+a^2}$.

Problem 4. Let X be an infinite dimensional normed space. Show that the unit ball in the dual space X^* is not compact. (Do NOT quote that unit ball in any infinite dimensional normed space is non compact)

Problem 5. Let $(X, \|\cdot\|)$ be a Banach space and let p be a functional on X satisfying the following condition
(i) $p(x) \geq 0$ for all $x \in X$;
(ii) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha > 0$;
(iii) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ for all $x_1, x_2 \in X$;
(iv) If $x_n \to x_\infty$ in X, then $p(x) \leq \lim p(x_n)$.

Show that there is a $M > 0$ such that $p(x) \leq M \cdot \|x\|$ for all $x \in X$.

Problem 6. Let $e_n = (0, \cdots, 0, 1, 0, \cdots)$ (1 at the n-th entry), $n = 1, 2, \cdots$, be a basis of real Hilbert space l^2. Define linear operator A on l^2 by

$$A(e_n) = \sum_{k=1}^{\infty} \alpha_{nk} e_k,$$

where $\alpha_{nk} \in \mathbb{R}$ satisfying $\sum_{n,k=1}^{\infty} |\alpha_{nk}|^2 < \infty$. Prove that for any bounded sequence $f_n = (f_{n1}, f_{n2}, \cdots)$, $n = 1, 2, \cdots$, in l^2, there is a subsequence of $\{A(f_n)\}_{n=1}^{\infty}$ which converges in l^2.

Problem 7. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1 + x^2} dx.$$

Problem 8. Let $\{f_n\}$ be a sequence of holomorphic functions on the open unit disc D which converges to f_∞ uniformly on every compact subset of D. Suppose that each f_n is one-to-one. Prove that f_∞ is also one-to-one.

Problem 9. Let A be the set of all continuous functions on the closed unit disc $\bar{D} = \{z \in \mathbb{C}, |z| \leq 1\}$ which are holomorphic on $\{z \in \mathbb{C}, |z| < 1\}$. Define for any $f \in A$

$$\|f\| = \sup_{z \in D} |f(z)|.$$

It is known that $(A, \|\cdot\|)$ is a normed space. Show that $(A, \|\cdot\|)$ is actually a Banach space.

Qualifying Exam Analysis Winter 2008
Solutions

Problem 1. Let $E \subset [0,1]$ be a Lebesque measurable set with $m(E) > 0$. Show that there is a subset $F \subset E$ which is not Lebesque measurable.

Solution to Problem 1. The same construction as show that here is a non-Lebesque measurable set in $[0,1]$. I.e., define an equivalence relation $\sim$ on E by $x \sim y$ for $x, y \in E$ if $x - y$ is a rational number, then using Zorn lemma to construct a subset F of E by choosing one element from each equivalence class.

Problem 2. Let $\{f_n\}$ be a sequence in $L^1(X,\mu)$ satisfying

$$\lim_{n\to\infty} \int_S |f_n| d\mu = 0$$

for all measurable subsets $S \subset X$. Assume there is a $M > 0$ such that $|f_n(x)| \leq M$ for all n and $x \in X$. Show that, for any $f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \int_X |ff_n| d\mu = 0.$$

Solution to Problem 2. Given any $\epsilon > 0$ there is a simple function $s = \sum_{i=1}^m c_i \chi_{E_i}$ such that $\int_X |f - s| d\mu < \epsilon$. We estimate

$$\int_X |ff_n| d\mu \leq \int_X |f - s| \cdot |f_n| d\mu + \int_X |sf_n| d\mu \leq M\epsilon + \int_X |sf_n| d\mu$$

Note that

$$\lim_{n\to\infty} \int_X |sf_n| d\mu = \lim_{n\to\infty} \int_X |(\sum_{i=1}^m c_i \chi_{E_i}) f_n| d\mu$$

$$\leq \sum_{i=1}^m |c_i| \cdot \lim_{n\to\infty} \int_X |\chi_{E_i} f_n| d\mu = \sum_{i=1}^m |c_i| \cdot \lim_{n\to\infty} \int_{\chi_{E_i}} |f_n| d\mu$$

$$= 0.$$

Hence we have proved that

$$\lim_{n\to\infty}\int_X |ff_n|d\mu \le \epsilon.$$

and

$$\lim_{n\to\infty}\int_X |ff_n|d\mu = 0.$$

Problem 3. Show the following equalities ($a > 0$).
(i) $\int_0^\infty e^{-nx}\sin(ax)dx = \frac{a}{n^2+a^2}$ for any $n > 0$.

(ii) $\int_0^\infty \frac{\sin(ax)}{e^x-1}dx = \sum_{n=1}^\infty \frac{a}{n^2+a^2}$.

Solution to Problem 3. (i) This is a simple integration by parts twice exercise.

(ii) Note

$$\frac{\sin(ax)}{e^x-1} = e^{-x}\cdot\frac{\sin(ax)}{1-e^{-x}} = \sum_{n=1}^\infty e^{-nx}\sin(ax).$$

Since

$$|\sum_{n=1}^k e^{-nx}\sin(ax)| \le \sum_{n=1}^\infty e^{-nx}|\sin(ax)| \le \sum_{n=1}^\infty e^{-nx}|ax| = \frac{|ax|}{e^x-1},$$

and $\int_0^\infty \frac{|ax|}{e^x-1}dx < \infty$, we have

$$\int_0^\infty \frac{\sin(ax)}{e^x-1}dx = \int_0^\infty \sum_{n=1}^\infty e^{-nx}\sin(ax)dx$$
$$= \sum_{n=1}^\infty \int_0^\infty e^{-nx}\sin(ax)dx = \sum_{n=1}^\infty \frac{a}{n^2+a^2}.$$

Problem 4. Let X be an infinite dimensional normed space. Show that the unit ball in the dual space X^* is not compact. (Do NOT quote that unit ball in any infinite dimensional normed space is non compact)

Solution to Problem 4. Let $x_1, x_2, \cdots$ be a infinite sequence of linear independent elements with $\|x_n\| = 1$. We define a linear functional f_n on $E_n = span\{x_1, \cdots, x_n\}$ by defining $f(x_i) = \delta_{in}$. Since E_n is finite dimensional, f_n is a bounded linear functional on $(E_n, \|\cdot\|)$. We can scale f_n to a new functional g_n with $\|g_n\| = 1$. There is a $y_n \in E_n$ with $\|y_n\| = 1$ and $g_n(y_n) = 1$. Also note $g_n(E_{n-1}) = 0$.

By the Hahn-Banach theorem there is a linear functional G_n on X with norm 1 which extends g_n. We claim that $\{G_n\}$ has no convergent subsequence. For any two element G_n and G_m with $n < m$ we compute

$$\|G_n - G_m\| \geq |(G_n - G_m)(y_n)| = |1 - 0| = 1.$$

Problem 5. Let $(X, \|\cdot\|)$ be a Banach space and let p be a functional on X satisfying the following condition
(i) $p(x) \geq 0$ for all $x \in X$;
(ii) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha > 0$;
(iii) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ for all $x_1, x_2 \in X$;
(iv) If $x_n \to x_\infty$ in X, then $p(x) \leq \lim p(x_n)$.

Show that there is a $M > 0$ such that $p(x) \leq M \cdot \|x\|$ for all $x \in X$.

Solution to Problem 5. Given any $k \in \mathbb{N}$ we define $D_k = \{x \in X, p(x) \leq k\}$. It is clear that $\cup_k D_k = X$. By (iv) D_k is a closed set. Since X is complete, by Baire category theorem there is an open ball $B(x_0, r) \subset D_{k_0}$ for some k_0. For any y with $\|y\| \leq r$, we compute by (iii)

$$p(y) \leq p(x_0 + y) + p(-x_0) \leq k_0 + |p(-x_0)| \doteq M_1.$$

Hence for any $x \neq 0$ we have

$$p(x) = p(\frac{\|x\|}{r} \cdot \frac{rx}{\|x\|}) = \frac{\|x\|}{r} \cdot p(\frac{rx}{\|x\|}) \leq \frac{M_1}{r}\|x\|.$$

Problem 6. Let $e_n = (0, \cdots, 0, 1, 0, \cdots)$ (1 at the n-th entry), $n = 1, 2, \cdots$, be a basis of real Hilbert space l^2. Define linear operator A on l^2 by

$$A(e_n) = \sum_{k=1}^{\infty} \alpha_{nk} e_k,$$

where $\alpha_{nk} \in \mathbb{R}$ satisfying $\sum_{n,k=1}^{\infty} |\alpha_{nk}|^2 < \infty$. Prove that for any bounded sequence $f_n = (f_{n1}, f_{n2}, \cdots)$, $n = 1, 2, \cdots$, in l^2, there is a subsequence of $\{A(f_n)\}_{n=1}^{\infty}$ which converges in l^2.

Solution to Problem 6. Since fix a m, the sequence $\{f_{nm}\}$ is bounded, by a diagonalization argument we can find a subsequence of $\{f_n\}$ (still denote by the same sequence) such that for each m we have $f_{nm} \to f_{\infty m}$. We define $f_\infty = (f_{\infty 1}, f_{\infty 1}, \cdots)$. It follows from $\sum_{k=1}^{m} f_{nk}^2 \leq \|f_n\| \leq C$ for all n, taking the limit of $n \to \infty$ we get $\sum_{k=1}^{m} f_{\infty k}^2 \leq C$ for all m, hence $f_\infty \in l^2$. Note $|f_{nm}|^2 \leq C$.

Now we show that $A(f_n) \to A(f_\infty)$. Given any $\epsilon > 0$, we choose N such that $\sum_{n\geq N \text{ or } k\geq N}^{\infty} |\alpha_{nk}|^2 < \epsilon^2$. We compute

$$\|A(f_n) - A(f_\infty)\|^2 = \sum_{l=1}^{\infty} |\sum_{m=1}^{\infty} (\alpha_{ml}(f_{nm} - f_{\infty m})|^2$$
$$= \sum_{l=1}^{\infty} |\sum_{m=1}^{N-1} (\alpha_{ml}(f_{nm} - f_{\infty m})|^2 + \sum_{l=1}^{\infty} |\sum_{m=N}^{\infty} (\alpha_{ml}(f_{nm} - f_{\infty m})|^2$$
$$\leq \sum_{l=1}^{\infty} |\sum_{m=1}^{N-1} (\alpha_{ml}(f_{nm} - f_{\infty m})|^2 + 4C \sum_{l=1}^{\infty} \sum_{m=N}^{\infty} |\alpha_{ml}|^2$$
$$\leq \sum_{l=1}^{\infty} |\sum_{m=1}^{N-1} (\alpha_{ml}(f_{nm} - f_{\infty m})|^2 + 4C\epsilon^2.$$

However for the given N we can find n_0 such that for $n \geq n_0$ and $1 \leq m \leq N$ we have $|f_{nm} - f_{\infty m}| \leq \epsilon$. Hence we have for $n \geq n_0$

$$\|A(f_n) - A(f_\infty)\|^2 \leq \sum_{l=1}^{\infty} \sum_{m=1}^{N-1} |\alpha_{ml}|^2 \epsilon^2 + 4C\epsilon^2$$
$$\leq (\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} |\alpha_{ml}|^2 + 4C)\epsilon^2.$$

Hence $A(f_n)$ converges to $A(f_\infty)$ in l^2.

Problem 7. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2} dx.$$

Solution to Problem 7. Consider the counter-clockwise loop $\gamma = ([-R, R] \times \{0\}) \cup (\{R\} \times [0, R]) \cup ([-R, R] \times \{R\}) \cup (\{-R\} \times [0, R])$. It is easy to check that

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2} dx = \text{Re} \lim_{R\to\infty} \int_\gamma \frac{e^{2z\sqrt{-1}}}{1+z^2} dz$$

Since $\frac{e^{2z\sqrt{-1}}}{1+z^2}$ has one pole $z = i$ inside the domain enclosed by γ, we get

$$\int_\gamma \frac{e^{2z\sqrt{-1}}}{1+z^2} dz = 2\pi\sqrt{-1} \left. \frac{e^{2z\sqrt{-1}}}{2z} \right|_{z=\sqrt{-1}} = \pi e^{-2}.$$

Hence $\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2} dx = \pi e^{-2}$.

Problem 8. Let $\{f_n\}$ be a sequence of holomorphic functions on the open unit disc D which converges to f_∞ uniformly on every compact subset of D. Suppose that each f_n is one-to-one. Prove that f_∞ is also one-to-one.

Solution to Problem 8. It is know that f_∞ is a holomorphic functions on D. It f_∞ is not one-to-one. Choose a closed simple curve γ in D such that $z_1 \neq z_2$ are the only two points such that $f_\infty(z_1) = f_\infty(z_2) = c$ for some c. We may assume $\frac{1}{f(z)-c}$ has simple pole at z_1 and z_2. Otherwise we replace γ by a small circle around the z_i which is not the simple pole. Now we consider index

$$\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f'(z)}{f(z)-c} dz = 2.$$

By Theorem 10.28 on p.214 in [Rudin] we have

$$\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f'(z)}{f(z)-c} dz = \lim_{n\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f_n'(z)}{f_n(z)-c} dz.$$

However the index $\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f_n'(z)}{f_n(z)-c} dz$ is either 1 or 0 depending on $f_n(z)$ has solution in the domain enclosed by γ or not.

Problem 9. Let A be the set of all continuous functions on the closed unit disc $\bar{D} = \{z \in \mathbb{C}, |z| \leq 1\}$ which are holomorphic on $\{z \in \mathbb{C}, |z| < 1\}$. Define for any $f \in A$

$$\|f\| = \sup_{z\in D} |f(z)|.$$

It is known that $(A, \|\cdot\|)$ is a normed space. Show that $(A, \|\cdot\|)$ is actually a Banach space.

Solution to Problem 9. Is is enough to prove that $(A, \|\cdot\|)$ is complete. Note $(A, \|\cdot\|)$ is a subspace of $(C(\bar{D}, \mathbb{C}), \|\cdot\|)$ which is a complete normed space. Hence for any Cauchy sequence $f_n \in A$ there is a $f_\infty \in C(\bar{D}, \mathbb{C})$ such that $\|f_n - f_\infty\| \to 0$. By Theorem 10.28 on p.214 in [Rudin] we have $f_\infty \in A$. Hence $(A, \|\cdot\|)$ is a Banach space.