ANALYSIS QUALIFYING EXAM FOR WINTER 2008

Problem 0.1. Let $f\colon [0,1] \to \mathbb{R}$ be integrable, and suppose that $\lim_{x\to 1} f(x) = 0$. Prove that $\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = 0$.

Problem 0.2. Let $g\colon [0,1] \to \mathbb{C}$ be absolutely continuous and satisfy $g(0) = 0$, and suppose $1 \le p \le \infty$. Prove that $\|g\|_\infty \le \|g'\|_p$.

Problem 0.3. Let A be a measurable subset of $\mathbb{R}$ with $0 < m(A) < \infty$. Show that there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there are points $x, y \in A$ with $|x - y| = \varepsilon$.

(Hint: Determine $\lim_{\varepsilon\to 0} m(A \cap (A + \varepsilon))$, by expressing $m(A \cap (A + \varepsilon))$ as an integral over A.)

Problem 0.4. Let E be a Banach space. Let $\mathrm{Lip}([0,1], E)$ be the vector space of all functionss from $[0,1]$ to E such that the quantity

$$\|f\| = |f(0)| + \sup\left(\left\{\frac{\|f(s) - f(t)\|}{|s-t|} : s, t \in [0,1] \text{ with } s \neq t\right\}\right)$$

is finite. Then $\mathrm{Lip}([0,1], E)$ is a vector space (with the pointwise operations), and $\|\cdot\|$ is norm on it. (You need not prove these facts.) Prove that $\mathrm{Lip}([0,1], E)$ is complete.

Problem 0.5. Let $V = \mathrm{span}(\{e_n \colon n \in \mathbb{Z}\})$ be the linear subspace of $L^2([a,b])$ spanned by the functions $e_n(x) = e^{2\pi i n x}$ for $n \in \mathbb{Z}$. Denote by $V^\perp$ the orthogonal complement of V in $L^2([a,b])$. Prove that if $b - a < 1$ then $V^\perp = 0$.

Problem 0.6. Let $\Omega_1 \subset \Omega_2$ be regions in $\mathbb{C}$, with $\Omega_1 \neq \varnothing$ and $\overline{\Omega_2}$ compact. For $j = 1, 2$ let E_j be the Banach space defined by

$$E_j = \left\{f \in C(\overline{\Omega_j}) \colon f|_{\Omega_j} \text{ is holomorphic}\right\}.$$

Further let $M_j \subset E_j$ be a closed subspace. Suppose that the restriction map $f \mapsto f|_{\overline{\Omega_1}}$, from M_2 to M_1, is surjective. Prove that there is a bounded linear map $T\colon M_1 \to M_2$ such that $T(f)|_{\overline{\Omega_1}} = f$ for all $f \in M_1$.

Problem 0.7. Let $\Omega \subset \mathbb{C}$ be connected and open, and let $f\colon \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Suppose that $|f|$ has a local minimum at $a \in \Omega$. Prove that $f(a) = 0$.

Problem 0.8. Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{1 + x^2}\,dx.$$

Problem 0.9. Let $X = \{0\} \cup \left\{\frac{1}{n} \colon n \in \mathbb{N}\right\}$. Let $f\colon \mathbb{C} \setminus X \to \mathbb{C}$ be a holomorphic function which is bounded on $\{z \in \mathbb{C} \colon |z| < 2\} \setminus X$. Prove that f is the restriction to $\mathbb{C} \setminus X$ of an entire function.

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ANALYSIS QUALIFYING EXAM SOLUTIONS FOR WINTER 2008

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Problem 0.1. Let $f\colon [0,1] \to \mathbb{R}$ be integrable, and suppose that $\lim_{x \to 1} f(x) = 0$. Prove that $\lim_{n \to \infty} n \int_0^1 x^n f(x)\, dx = 0$.

Harder variation: Suppose merely that $a = \lim_{x \to 1} f(x)$ exists, and determine $\lim_{n \to \infty} n \int_0^1 x^n f(x)\, dx$. (The limit is a.)

This is based on a Berkeley prelim problem from Fall 1981. Measure theory is not assumed; the problem had f continuous on $[0,1]$, and asked that one find the limit.

Solution. Let $\varepsilon > 0$. It suffices to prove that

$$\limsup_{n \to \infty} \left| n \int_0^1 x^n f(x)\, dx \right| \leq \varepsilon.$$

Choose $r \in (0,1)$ such that $|f(x)| < \varepsilon$ for all $x \in [r,1]$. Since $r < 1$, we have $\lim_{n \to \infty} n r^n = 0$, so there is M such that $n r^n \leq M$ for all $n \in \mathbb{N}$. For $x \in [0,r)$, then we have $|n x^n f(x)| \leq M |f(x)|$. Applying the Dominated Convergence Theorem on $[0,r)$ with $M|f|$ as the dominating function, and using $\lim_{n \to \infty} n x^n = 0$ for $x \in [0,r)$, we get

$$\lim_{n \to \infty} \left| n \int_0^r x^n f(x)\, dx \right| = 0.$$

Furthermore,

$$\left| n \int_r^1 x^n f(x)\, dx \right| \leq n \int_r^1 x^n |f(x)|\, dx \leq n \int_r^1 x^n \varepsilon\, dx = \frac{n\varepsilon(1 - r^{n+1})}{n+1},$$

whence

$$\limsup_{n \to \infty} \left| n \int_r^1 x^n f(x)\, dx \right| \leq \lim_{n \to \infty} \frac{n\varepsilon(1 - r^{n+1})}{n+1} = \varepsilon.$$

Thus,

$$\begin{aligned} \limsup_{n \to \infty} \left| n \int_0^1 x^n f(x)\, dx \right| &\leq \limsup_{n \to \infty} \left| n \int_0^r x^n f(x)\, dx \right| + \limsup_{n \to \infty} \left| n \int_r^1 x^n f(x)\, dx \right| \\ &\leq 0 + \varepsilon = \varepsilon. \end{aligned}$$

This completes the solution. □

Problem 0.2. Let $g\colon [0,1] \to \mathbb{C}$ be absolutely continuous and satisfy $g(0) = 0$, and suppose $1 \leq p \leq \infty$. Prove that $\|g\|_\infty \leq \|g'\|_p$.

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Still need to check if anything like this has been used before.

This problem is based on a Berkeley prelim problem from Fall 1977. The Berkeley prelim does not require measure theory. In the original problem, g is C^2, $p = 2$, and the norm notation is not used.

Solution. Since g is absolutely continuous, for all $x \in [0, 1]$ we have

$$g(x) = g(x) - g(0) = \int_0^x g' \, dm = \int_0^1 g' \chi_{[0,x]} \, dm.$$

With $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality then gives

$$|g(x)| \leq \|g'\|_p \|\chi_{[0,x]}\|_q \leq \|g'\|_p.$$

Take the sup over all $x \in [0, 1]$. □

Problem 0.3. Let A be a measurable subset of $\mathbb{R}$ with $0 < m(A) < \infty$. Show that there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there are points $x, y \in A$ with $|x - y| = \varepsilon$.

(Hint: Determine $\lim_{\varepsilon \to 0} m(A \cap (A + \varepsilon))$, by expressing $m(A \cap (A + \varepsilon))$ as an integral over A.)

From the Purdue graduate examination in measure theory, August 1996. I have rearranged the hint.

Solution. For $t \in \mathbb{R}$, set $f(t) = m(A \cap (A + t))$. We claim that f is continuous. Since $\chi_A \in L^1(\mathbb{R})$, we know (from a theorem in Rudin) that $t \mapsto \chi_{A+t}$, which is the translate of χ_A by t, is continuous. (If this has been forgotten, it is not hard to prove. If $g \in C_c(\mathbb{R})$, then the map sending t to the translate of g by t is continuous, essentially because g is uniformly continuous and its support has a neighborhood with finite measure. Now use a 3ε argument and density of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$.) Now for $s, t \in \mathbb{R}$, we have

$$\begin{aligned} |f(s) - f(t)| &= \left| \int_A \chi_{(A+s)} \, dm - \int_A \chi_{(A+t)} \, dm \right| \\ &\leq \int_A |\chi_{(A+s)} - \chi_{(A+t)}| \, dm \leq \|\chi_{(A+s)} - \chi_{(A+t)}\|_1. \end{aligned}$$

This implies that f is continuous.

We have $f(0) = m(A) > 0$. Choose $\varepsilon_0 > 0$ such that $f(t) > 0$ for $|t| < \varepsilon_0$. Then, in particular, $0 < \varepsilon < \varepsilon_0$ implies $f(\varepsilon) > 0$, so $A \cap (A + \varepsilon) \neq \varnothing$. If $y \in A \cap (A + \varepsilon)$, then $x = y - \varepsilon \in A$, so $x, y \in A$ and satisfy $|x - y| = \varepsilon$. □

Problem 0.4. Let E be a Banach space. Let $\mathrm{Lip}([0, 1], E)$ be the vector space of all functionss from $[0, 1]$ to E such that the quantity

$$\|f\| = |f(0)| + \sup\left(\left\{ \frac{\|f(s) - f(t)\|}{|s - t|} : s, t \in [0, 1] \text{ with } s \neq t \right\} \right)$$

is finite. Then $\mathrm{Lip}([0, 1], E)$ is a vector space (with the pointwise operations), and $\| \cdot \|$ is norm on it. (You need not prove these facts.) Prove that $\mathrm{Lip}([0, 1], E)$ is complete.

At least half the credit will be lost if the last step (proving that $\|f_n - f\|$ actually converges to zero) is omitted.

Solution. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Lip}([0,1], E)$. Then

$$\|f_m(0) - f_n(0)\| \leq \|f_m - f_n\|,$$

so $(f_n(0))_{n\in\mathbb{N}}$ is Cauchy in E. For every $t \in (0,1]$, we have, taking $s = 0$ in the supremum in the definition of $\|\cdot\|$,

$$t^{-1}\|f_m(t) - f_m(0) - f_n(t) + f_n(0)\| \leq \|f_m - f_n\|,$$

so

$$\|f_m(t) - f_n(t)\| \leq t\|f_m - f_n\| + \|f_m(0) - f_n(0)\|.$$

Therefore the sequence $(f_n(t))_{n\in\mathbb{N}}$ is also Cauchy in E. It follows that, for every $t \in [0,1]$, the sequence $(f_n(t))_{n\in\mathbb{N}}$ has a limit $f(t) \in E$.

We claim that $f \in \mathrm{Lip}([0,1], E)$. Certainly $\|f(0)\| < \infty$. Also, for $s, t \in [0,1]$ with $s \neq t$ we have

$$\frac{\|f(s) - f(t)\|}{|s-t|} = \lim_{n\to\infty} \frac{\|f_n(s) - f_n(t)\|}{|s-t|} \leq \limsup_{n\to\infty} \|f_n\|,$$

and the last expression is finite because $(\|f_n\|)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. This implies the claim.

We finish by proving that $\lim_{n\to\infty} \|f_n - f\| = 0$. Let $\varepsilon > 0$, and choose N such that $m, n \geq N$ imply $\|f_m - f_n\| < \frac{1}{3}\varepsilon$. For any $n \geq N$, we have

$$\|(f_n - f)(0)\| = \lim_{m\to\infty} \|f_n(0) - f_m(0)\| \leq \limsup_{m\to\infty} \|f_m - f_n\| \leq \tfrac{1}{3}\varepsilon.$$

For $s, t \in [0,1]$ and $n \geq N$, we have

$$\begin{aligned}\frac{\|f_n(s) - f(s) - f_n(t) + f(t)\|}{|s-t|} &= \lim_{m\to\infty} \frac{\|f_n(s) - f_m(s) - f_n(t) + f_m(t)\|}{|s-t|} \\ &\leq \limsup_{m\to\infty} \|f_m - f_n\| \leq \tfrac{1}{3}\varepsilon.\end{aligned}$$

So

$$\sup\left(\left\{\frac{\|(f_n - f)(s) - (f_n - f)(t)\|}{|s-t|} : s, t \in [0,1] \text{ with } s \neq t\right\}\right) \leq \tfrac{1}{3}\varepsilon.$$

Adding the estimate on $\|(f_n - f)(0)\|$, we see that $n \geq N$ implies $\|f_n - f\| \leq \frac{2}{3}\varepsilon < \varepsilon$. This proves the claim. Thus, $\lim_{n\to\infty} f_n = f$, and we have finished the proof that $\mathrm{Lip}([0,1], E)$ is complete. □

Problem 0.5. Let $V = \mathrm{span}(\{e_n : n \in \mathbb{Z}\})$ be the linear subspace of $L^2([a,b])$ spanned by the functions $e_n(x) = e^{2\pi i n x}$ for $n \in \mathbb{Z}$. Denote by $V^\perp$ the orthogonal complement of V in $L^2([a,b])$. Prove that if $b - a < 1$ then $V^\perp = 0$.

Is it too easy? If so, can add a second part: Prove that if $b - a > 1$ then $V^\perp \neq 0$.

It is assumed that the students may use without proof the fact that the functions e_n form an orthonormal basis of $L^2([0,1])$.

Solution. Set $f_n(x) = \exp(2\pi i n x)$ for $n \in \mathbb{Z}$ and $x \in [0,1]$. These form an orthonormal basis of $L^2([0,1])$. Translating everything by a, we see that the functions

$$x \mapsto \exp(2\pi i n(x-a)) = \exp(-2\pi i n a)\exp(2\pi i n x),$$

for $n \in \mathbb{Z}$, form an orthonormal basis of $L^2([a, a+1])$. We may multiply the basis elements by scalars of absolute value 1 without changing the fact that they form an orthonormal basis, so the functions e_n, for $n \in \mathbb{Z}$, form an orthonormal basis of $L^2([a, a+1])$.

Now let $f \in V^{\perp} \subset L^2([a,b])$. Let $g \in L^2([a,\ a+1])$ be the extension of f to $[0,1]$ by zero, that is,

$$g(x) = \begin{cases} f(x) & a \leq x \leq b \\ 0 & b < x \leq a+1. \end{cases}$$

Then for all $n \in \mathbb{Z}$ we have

$$0 = \int_a^b f(x)e_n(x)\,dx = \int_a^{a+1} g(x)e_n(x)\,dx.$$

This implies that $g = 0$ and, therefore, $f = 0$. □

Alternate solution. The space V is a subalgebra of $C([a,b])$ (because $e_m e_n = e_{m+n}$) which is closed under complex conjugation (because $\overline{e_n(x)} = e_{-n}(x)$). Furthermore, since $b - a < 1$, we have $x \neq y$ implies $e_1(x) \neq e_1(y)$. Therefore V separates the points of $[a,b]$. So V is dense in $C([a,b])$ by the Stone-Weierstrass Theorem. Since $\|f\|_2 \leq (b-a)^{1/2}\|f\|_\infty$ for any $f \in C([0,b])$, it follows that V is dense in $C([a,b])$ in $\|\cdot\|_2$ as well. We know that $C([a,b])$ is dense in $L^2([a,b])$ in $\|\cdot\|_2$, so it follows that V is dense in $L^2([a,b])$. Therefore $V^{\perp} = 0$. □

Problem 0.6. Let $\Omega_1 \subset \Omega_2$ be regions in $\mathbb{C}$, with $\Omega_1 \neq \varnothing$ and $\overline{\Omega_2}$ compact. For $j = 1, 2$ let E_j be the Banach space defined by

$$E_j = \left\{ f \in C\big(\overline{\Omega_j}\big) \colon f|_{\Omega_j} \text{ is holomorphic} \right\}.$$

Further let $M_j \subset E_j$ be a closed subspace. Suppose that the restriction map $f \mapsto f|_{\overline{\Omega_1}}$, from M_2 to M_1, is surjective. Prove that there is a bounded linear map $T \colon M_1 \to M_2$ such that $T(f)|_{\overline{\Omega_1}} = f$ for all $f \in M_1$.

Example of such a situation: Set

$$\Omega_1 = \{z \in \mathbb{C} \colon 1 < |z| < 2\} \quad \text{and} \quad \Omega_2 = \{z \in \mathbb{C} \colon |z| < 2\}.$$

Let M_1 be any closed subspace of E_1, and take

$$M_2 = \left\{ f|_{\overline{\Omega_1}} \colon f \in M_2 \right\}.$$

The restriction map is isometric, so the image is closed.

It is not clear that there are more interesting examples.

Variation: Prove that for $z \in \overline{\Omega_2}$, there is a Borel measure μ on $\overline{\Omega_1}$ such that $f(z) = \int_{\overline{\Omega_1}} f\,d\mu$ for all $f \in E_2$. The proof requires two additional steps: the Hahn-Banach theorem and the Riesz Representation Theorem. It can be slightly simplified by requiring that $M_1 = E_1$.

Solution. Let $S \colon M_2 \to M_1$ be the restriction map. The hypotheses state that S is surjective, and it is obviously continuous. The Open Mapping Theorem now implies that S is an open mapping.

We claim that S is injective. Let $f \in M_2$ and assume that $f|_{\overline{\Omega_1}} = 0$. Then $f|_{\Omega_2}$ is a holomorphic function which vanishes on a subset of Ω_2 (namely Ω_1) which contains a limit point. Therefore $f|_{\Omega_2} = 0$. It follows that f vanishes on $\overline{\Omega_2}$. This proves the claim.

The claim implies that $T = S^{-1}$ exists. Since S is open, T is continuous. Thus, T is the required map. □

Problem 0.7. Let $\Omega \subset \mathbb{C}$ be connected and open, and let $f \colon \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Suppose that $|f|$ has a local minimum at $a \in \Omega$. Prove that $f(a) = 0$.

Solution. Suppose $f(a) \neq 0$. Choose $r > 0$ so small that the set $D = \{z \in \mathbb{C} \colon |z - a| < r\}$ is contained in Ω and $f(z) \neq 0$ for $z \in D$. Then $g = 1/(f|_D)$ is a holomorphic function on D, and $|g|$ has a local maximum at $a \in D$. The Maximum Modulus Theorem implies that g is constant. Therefore $f|_D$ is constant. Since Ω is connected, it follows that f is constant. $\square$

Problem 0.8. Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2}\, dx.$$

Solution. Set

$$f(z) = \frac{e^{2iz}}{1+z^2}$$

for $z \in \mathbb{C} \setminus \{\omega \in \mathbb{C} \colon \omega^2 = -1\}$. For $R > 0$, define paths $\gamma_R \colon [-R, R] \to \mathbb{C}$ and $\eta_R \colon [0, \pi] \to \mathbb{C}$ by $\gamma_R(t) = t$ and $\eta_R(t) = e^{it}$.

It is easy to check that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = \int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2}\, dx.$$

We claim that

$$\lim_{R\to\infty} \int_{\eta_R} f(z)\, dz = 0.$$

Begin by writing

$$\int_{\eta_R} f(z)\, dz = \int_0^{\pi} \frac{\exp(2iRe^{it})Rie^{it}}{1+R^2e^{2it}}\, dt.$$

Set

$$g_R(t) = \frac{\exp(2iRe^{it})Rie^{it}}{1+R^2e^{2it}}.$$

For $R \geq 2$, we have $|1 + R^2e^{2it}| \geq R^2 - 1$. So, using $\mathrm{Im}(e^{it}) \geq 0$ for $t \in [0, \pi]$, we have

$$|g_R(t)| \leq \exp\left(\mathrm{Re}(2iRe^{it})\right)\left(\frac{R}{R^2-1}\right) \leq \frac{R}{R^2-1}.$$

This shows that the functions g_R converge uniformly to zero. The claim follows.

For $R > 1$, we can use the Residue Theorem to calculate $\int_{\gamma_R} f(z)\, dz + \int_{\eta_R} f(z)\, dz$. This is the integral around a closed path. There are two poles, at the numbers ω such that $\omega^2 = -1$, that is, at $\pm i$. The pole at $-i$ is in the lower half plane, and has winding number zero. The pole at i is in the upper half plane, and has winding number one. This pole is simple, so we can evaluate the residue as follows (using the factorization $z^2 + 1 = (z - i)(z + i)$):

$$\mathrm{Res}(f; i) = \lim_{z\to i}(z-i)f(z) = \frac{\exp(2i^2)}{i+i} = -\frac{i}{2e^2}.$$

Therefore, for any $R > 1$,

$$\int_{\gamma_R} f(z)\, dz + \int_{\eta_R} f(z)\, dz = 2\pi i\left(-\frac{i}{2e}\right) = \frac{\pi}{e^2}.$$

Letting $R \to \infty$, we see that the right hand side is also the value of

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2}\, dx.$$

This completes the solution. $\square$

In the proof of the claim, one can use the Dominated Convergence Theorem instead of uniform convergence of the integrands. This requires that one consider an arbitrary sequence $(R_n)_{n\in\mathbb{N}}$ in $[2,\infty)$ such that $\lim_{n\to\infty} R_n = \infty$.

Since we know ahead of time that

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2}\, dx$$

exists, it actually suffices to do the main part of the proof for only one sequence $(R_n)_{n\in\mathbb{N}}$ converging to ∞. A proof correctly written that way will be clear. Some credit will be lost for anything that looks like the application of the Dominated Convergence Theorem using $R \to \infty$ through general real numbers.

Problem 0.9. Let $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Let $f : \mathbb{C} \setminus X \to \mathbb{C}$ be a holomorphic function which is bounded on $\{z \in \mathbb{C} : |z| < 2\} \setminus X$. Prove that f is the restriction to $\mathbb{C} \setminus X$ of an entire function.

The Winter 2005 qualifying exam had the related problem: Prove that a bounded holomorphic function on $\mathbb{C} \setminus \mathbb{Z}$ is constant.

Solution. For each $n \in \mathbb{N}$, the function f has an isolated singularity at $\frac{1}{n}$. Since f is bounded on some deleted ball $B_\varepsilon\left(\frac{1}{n}\right) \setminus \left\{\frac{1}{n}\right\} \to \mathbb{C}$, it follows that this singularity is removable. Therefore there is a holomorphic function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $g|_{\mathbb{C}\setminus X} = f$. By continuity, g is bounded on $\{z \in \mathbb{C} :) < |z| < 2\}$. Now g has a removable singularity at 0. So there is an entire function h such that $h|_{\mathbb{C}\setminus\{0\}} = f$. $\square$