Analysis Qualifying Exam, Winter 2007

1. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on $[0,1]$, which is converging uniformly. Determine whether the limit below exists

$$\lim_{n\to\infty}\int_0^1 nf_n(t)2^{-nt}dt.$$

If yes, then find its value.

2. Give an example of a dense G_δ set $E\subset[0,1]$ such that $|E|=0$.

3. Let μ be a positive measure on X. Prove that

$$0<r<p<s<\infty \implies L^r(\mu)\cap L^s(\mu)\subset L^p(\mu).$$

4. Let $f_a(x)=\exp(-(x-a)^2/2)$. Prove that the linear combinations of $\{f_a : a\in\mathbb{R}\}$ are dense in $L^2(\mathbb{R})$.

5. Let μ be a positive σ-finite measure and $1\le p<\infty$. Let f be a measurable function. Prove that a multiplication operator $M_f : L^p(\mu)\to L^p(\mu)$, $M_f(g)=fg$, is bounded $\iff$ $||f||_\infty<\infty$.

6. Suppose that $f:[a,b]\to\mathbb{R}$ is AC (absolutely continuous). Prove that that there exist non-decreasing AC functions $f_1,f_2:[a,b]\to\mathbb{R}$ such that $f=f_1-f_2$.

7. Find a bounded linear functional L on $L^p(\mathbb{R})$, $1\le p<\infty$, such that

$$L(\chi_{[k,k+1]})=(1+|k|)^{-1} \qquad \text{for all } k\in\mathbb{Z}.$$

8. Let g be a meromorphic function on a simply connected domain Ω such that $g(z)\neq 0$ for $z\in\Omega$. Let γ^* be a positively oriented circular path in Ω which does not go through any of the poles of g. Prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(\xi)}{g(\xi)}d\xi = \sum_{\text{poles } p \text{ inside } \gamma} N(p),$$

where $N(p)$ is the order of a pole p.

9. Suppose that $f\in H(\Omega)$, a domain Ω contains a closed unit disc, $|f(z)|>2$ for $|z|=1$, and $f(0)=\sqrt{2}$. Must f have a zero in the unit disc?