Analysis Qualifying Examination

~~Spring~~ 2006 Winter

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let μ be a finite Borel measure on the the unit disk. Show that, for any $\epsilon > 0$, there is $0 < r < \epsilon$ such that

$$\mu(C_r) = 0,$$

where $C_r = \{z \in \mathbf{C} : |z| = r\}$.

2. Let $\{f_n\}$ be a sequence of measurable functions on X which converges to f in measure. Show that there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges to f almost everywhere on X.

3. Let $\Omega = \{z \in \mathbf{C} : \text{Re}z > 0, |z| < 2\}$. Suppose that $f \in H(\Omega)$ and $f \in C(S)$, where $S = \{z \in \mathbf{C} : \text{Re}z \geq 0, |z| \leq 1\}$. Show that

$$f(i/2) = \frac{1}{2\pi i} \int_{\partial(S)} \frac{f(z)}{z - i/2} dz,$$

where $\partial(S)$ is the positively oriented simple semi-circle.

4. Let f be a non-decreasing function on $[0,1]$. Show that $f' \in L([0,1])$ and

$$\int_0^1 f'(t)dt \leq f(1) - f(0).$$

5. Let X be a compact metric space and μ_n be a sequence of finite Borel measures on X. Suppose that there is a finite Borel measure μ on X such that

$$\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu$$

for all $f \in C(X)$. Show that there is $K > 0$ such that

$$|\mu_n|(X) \le K$$

for all n.

6. Prove that the closed unit ball of an infinite dimensional Hilbert space is not compact.

7. Let f be an entire function. Suppose that $f(1/z)$ has a pole of order k at zero. Show that $f(z)$ is a polynomial.

8. Evaluate

$$\int_{-\infty}^{\infty} \frac{3x^2}{1+x^4} dx$$

and justify your computation.

9. Let n be a positive integer and let a be a real number such that $a > e$. Show that the equation $e^z = az^n$ has n solutions (counting multiplicity) inside the unit circle.