ANALYSIS QUALIFYING EXAM WINTER 2005

1. Let $f, g \in L^1(\mathbf{R})$. Set $\tau_n(f)(x) = f(x-n)$. Prove that $\lim_{n\to\infty} \|\tau_n(f) + g\|_1$ exists.

2. Set $f(x) = \exp(-x^6)$ for $x \in \mathbf{R}$. Prove that the formula

$$g(z) = \int_{-\infty}^{\infty} f(x) e^{-izx}\, dx.$$

defines a continuous function on $\mathbf{C}$.

3. Evaluate

$$\int_0^3 \left(\int_x^3 \sin(y^2)\, dy \right) dx.$$

Be sure to justify all steps.

4. Let X be a compact metric space, let $M(X)$ be the Banach space of all complex Borel measures on X, and let $f\colon X \to \mathbf{C}$ be a bounded Borel function. Define a linear functional $\omega\colon M(X) \to \mathbf{C}$ by $\omega(\mu) = \int_X f\, d\mu$. Prove that ω is bounded, and find $\|\omega\|$.

5. Let E be a closed subspace of $C([0,2])$ such that for every $f \in C([0,1])$ there is $g \in E$ satisfying $g|_{[0,1]} = f$. Prove that there is a constant $M < \infty$ such that for every $f \in C([0,1])$ there is $g \in E$ satisfying $g|_{[0,1]} = f$ and $\|g\|_\infty \leq M\|f\|_\infty$.

6. Let $g_0, g_1, \ldots \in L^2(\mathbf{R})$, and suppose that $\|g_n\|_2 = \frac{1}{2^n}$ and that the Fourier transform $\widehat{g_n}$ vanishes outside $[n, n+1]$. Prove that $\sum_{n=0}^{\infty} g_n$ converges in $L^2(\mathbf{R})$ to some function $g \in L^2(\mathbf{R})$, and calculate $\|g\|_2$.

7. Let $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function. Suppose that $f|_{\mathbf{R}}$ is periodic. Prove that f is periodic, that is, that there is $a \in \mathbf{C} \setminus \{0\}$ such that $f(z+a) = f(z)$ for all $z \in \mathbf{C}$.

8. Let f be the meromorphic function on $\mathbf{C}$ given by

$$f(z) = \frac{e^z}{z} + \frac{17}{z+4}.$$

Evaluate $\int_\gamma f(z)\, dz$, where $\gamma\colon [0, 2\pi] \to \mathbf{C}$ is given by $\gamma(t) = 3e^{it}$.

9. Let $f\colon \mathbf{C} \setminus \mathbf{Z} \to \mathbf{C}$ be a bounded holomorphic function. Prove that f is constant.

SOLUTIONS TO ANALYSIS QUALIFYING EXAM WINTER 2005

Difficulty ratings are guesses, and are on a scale of 1 (easy) to 3 (hard).

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1. Let $f, g \in L^1(\mathbf{R})$. Set $\tau_n(f)(x) = f(x-n)$. Prove that $\lim_{n\to\infty} \|\tau_n(f) + g\|_1$ exists.

Comment: Difficulty rating: 2.

Solution: We prove that

$$\lim_{n\to\infty} \|\tau_n(f) + g\|_p = \|f\|_1 + \|g\|_1.$$

Let $\varepsilon > 0$. Choose $f_0, g_0 \in C_c(\mathbf{R})$ such that

$$\|f_0 - f\|_1 < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \|g_0 - g\|_1 < \tfrac{1}{2}\varepsilon.$$

Choose $N \in \mathbf{N}$ such that the supports of both f and g are contained in $[-N, N]$. Let $n \geq 2N$. Then

$$\begin{aligned}\|\tau_n(f_0) + g_0\|_1 &= \int_{-\infty}^{\infty} |\tau_n(f_0) + g_0|\, dm \\ &= \int_{n-N}^{n+N} |f_0(x-n)|\, dm(x) + \int_{-N}^{N} |g_0(x)|\, dm(x) \\ &= \int_{-N}^{N} |f_0|\, dm + \int_{-N}^{N} |g_0|\, dm = \|f_0\|_1 + \|g_0\|_1.\end{aligned}$$

(All that was really used here is that the supports of $\tau_n(f_0)$ and g_0 are disjoint.) We have

$$\big|\|f\|_1 - \|f_0\|_1\big| \leq \|f - f_0\|_1 < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \big|\|g\|_1 - \|g_0\|_1\big| \leq \|g - g_0\|_1 < \tfrac{1}{2}\varepsilon,$$

so

$$\begin{aligned}&\big|\|\tau_n(f_0) + g_0\|_1 - (\|f\|_1 + \|g\|_1)\big| \\ &\qquad = \big|(\|f_0\|_1 + \|g_0\|_1) - (\|f\|_1 + \|g\|_1)\big| < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$\lim_{n\to\infty} \|\tau_n(f) + g\|_1 = \|f\|_1 + \|g\|_1,$$

as desired. ∎

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2. Set $f(x) = \exp(-x^6)$ for $x \in \mathbf{R}$. Prove that the formula

$$g(z) = \int_{-\infty}^{\infty} f(x) e^{-izx}\, dx.$$

defines a continuous function on $\mathbf{C}$.

Comment: Difficulty rating: 2.

The final exam for Math 618 in Spring 1999 had the problem:

> Define $f(x) = \exp(-x^4)$ for $x \in \mathbf{R}$. Prove carefully that there is an entire function g whose restriction to $\mathbf{R}$ is the Fourier Transform $\widehat{f}$ of f.

This exam, and its solutions, were posted on my web site for the 2003–2004 course. In addition, attention was drawn to this problem because a corrected solution was posted. (I had forgotten to prove that g is continuous.)

This version of the problem, only asking for continuity, is much shorter, and its proof still contains an important idea: combining the Dominated Convergence Theorem with estimating something different ways on two different parts of its domain.

Solution: We are going to prove sequential convergence using the Dominated Convergence Theorem. This requires an estimate on the integrand. For $z \in \mathbf{C}$, we have

$$\begin{aligned} |f(x)e^{-izx}| &= |\exp(-x^6 - izx)| = \exp(\mathrm{Re}(-x^6 - izx)) \\ &= \exp(-x^6 + x\mathrm{Im}(z)) \leq \exp(-x^6 + |x| \cdot |z|). \end{aligned}$$

We claim that

$$|x| \cdot |z| \leq \tfrac{1}{2}x^6 + |z|(2|z| + 1).$$

We split the proof in two cases. If $|x| \leq 2|z| + 1$ then trivially $|x| \cdot |z| \leq |z|(2|z| + 1)$. If $|x| \geq 2|z| + 1$, then in particular $|x| \geq 1$, so

$$\tfrac{1}{2}x^6 \geq \tfrac{1}{2}x^2 \geq \tfrac{1}{2}|x|(2|z| + 1) = |x| \cdot |z| + \tfrac{1}{2} > |x| \cdot |z|.$$

This proves the claim.

It follows that

$$|f(x)e^{-izx}| \leq \exp(-\tfrac{1}{2}x^6)\exp(|z|(2|z| + 1))$$

for all $x \in \mathbf{R}$ and $z \in \mathbf{C}$.

We know that $\exp(-\frac{1}{2}x^6)$ is integrable on $\mathbf{R}$, and $\exp(|z|(2|z|+1))$ is a constant, so $x \mapsto |f(x)e^{-izx}|$ is integrable on $\mathbf{R}$, and $g(z)$ is defined for all z.

Now we prove continuity. Let $(z_n)_{n \in \mathbf{N}}$ be a sequence in $\mathbf{C}$ with $z_n \to z$. Let $M = \sup_{n \in \mathbf{N}} |z_n|$. Set $h_n(x) = f(x)e^{-iz_n x}$ for $n \in \mathbf{N}$, and set $h(x) = f(x)e^{-izx}$. Then $h_n(x) \to h(x)$ for all $x \in \mathbf{R}$. Also, with $C = \exp(M(2M + 1))$, we have $|h_n(x)| \leq C\exp(-\frac{1}{2}x^6)$, and $x \mapsto \exp(-\frac{1}{2}x^6)$ is integrable over $\mathbf{R}$. Therefore the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)e^{-iz_n x}\,dx = \int_{-\infty}^{\infty} f(x)e^{-izx}\,dx = g(z).$$

This proves continuity. ∎

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3. Evaluate

$$\int_0^3 \left(\int_x^3 \sin(y^2)\,dy \right) dx.$$

Be sure to justify all steps.

Comment: Difficulty rating: 1.

This might be very close to something that has been used recently—I didn't have the old exams available at the time I wrote it.

Solution: Let $T = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq y \leq 3\}$. Define $f : [0, 3] \times [0, 3] \to \mathbf{R}$ by

$$f(x, y) = \sin(y^2)\chi_T(x, y).$$

Then f is Borel because it is the product of a continuous function and the characteristic function of a closed set. Also, f is bounded. Therefore f is integrable on $[0, 3] \times [0, 3]$, and Fubini's Theorem for integrable functions applies. Using it at the second step, we have

$$\int_0^3 \left(\int_x^3 \sin(y^2)\, dy \right) dx = \int_0^3 \left(\int_0^3 f(x, y)\, dy \right) dx = \int_0^3 \left(\int_0^3 f(x, y)\, dx \right) dy$$
$$= \int_0^3 \int_0^y \sin(y^2)\, dx\, dy = \int_0^3 y \sin(y^2)\, dy.$$

The last integral is easily done via the substitution $u = y^2$, giving

$$\int_0^3 \left(\int_x^3 \sin(y^2)\, dy \right) dx = \tfrac{1}{2}\sin(3^2) - \tfrac{1}{2}\sin(0^2) = \tfrac{1}{2}\sin(9).$$

This completes the calculation. ∎

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4. Let X be a compact metric space, let $M(X)$ be the Banach space of all complex Borel measures on X, and let $f : X \to \mathbf{C}$ be a bounded Borel function. Define a linear functional $\omega : M(X) \to \mathbf{C}$ by $\omega(\mu) = \int_X f\, d\mu$. Prove that ω is bounded, and find $\|\omega\|$.

Comment: Difficulty rating: 1.

Solution: Set $M = \sup_{x \in X} |f(x)|$. We claim that $\|\omega\| = M$. In particular, this will show that $\|\omega\| < \infty$, so that ω is bounded.

We first prove that $\|\omega\| \leq M$. Let $\mu \in M(X)$. Use the Radon-Nikodym Theorem to write $\mu = g|\mu|$ for some Borel function $g \colon X \to \mathbf{C}$ such that $|g(x)| = 1$ for μ-almost every $x \in X$. Then

$$|\omega(\mu)| = \left| \int_X f\, d\mu \right| = \left| \int_X fg\, d|\mu| \right| \leq \int_X |fg|\, d|\mu| \leq M|\mu|(X) = M\|\mu\|.$$

Now we prove that $\|\omega\| \geq M$. Let $\varepsilon > 0$. Choose $x \in X$ such that $|f(x)| > M - \varepsilon$. Let μ be the point mass measure at x, that is,

$$\mu(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

Then $\|\mu\| = 1$ and $|\omega(\mu)| = |f(x)| > M - \varepsilon$. This shows that $\|\omega\| > M - \varepsilon$. ∎

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5. Let E be a closed subspace of $C([0, 2])$ such that for every $f \in C([0, 1])$ there is $g \in E$ satisfying $g|_{[0,1]} = f$. Prove that there is a constant $M < \infty$ such that for every $f \in C([0, 1])$ there is $g \in E$ satisfying $g|_{[0,1]} = f$ and $\|g\|_\infty \leq M\|f\|_\infty$.

Comment: Difficulty rating: 1.

Solution: Define $a\colon E \to C([0,1])$ by $a(f) = f|_{[0,1]}$. Then a is a surjective linear map between Banach spaces, and a is continuous because $\|a\| \leq 1$. The Open Mapping Theorem provides $\delta > 0$ such that the image in $C([0,1])$ of the open unit ball of E contains the open ball

$$B = \{f \in C([0,1])\colon \|f\|_\infty < \delta\}$$

in $C([0,1])$.

Now let $f \in C([0,1])$. We show that there is $g \in E$ with $\|g\|_\infty \leq \frac{2}{\delta}\|f\|_\infty$ such that $a(g) = f$. This will prove the result with $M = \frac{2}{\delta}$. If $f = 0$, take $g = 0$. Otherwise, set $\alpha = \dfrac{\delta}{2\|f\|_\infty}$. Then $\|\alpha f\|_\infty < \delta$. Therefore there is $g_0 \in E$ such that $\|g_0\|_\infty < 1$ and $a(g_0) = \alpha f$. So $g = \frac{1}{\alpha} g_0 \in E$ satisfies $\|g\|_\infty < \frac{2}{\delta}$ and $a(g) = f$, as desired. ∎

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6. Let $g_0, g_1, \ldots \in L^2(\mathbf{R})$, and suppose that $\|g_n\|_2 = \frac{1}{2^n}$ and that the Fourier transform $\widehat{g_n}$ vanishes outside $[n, n+1]$. Prove that $\sum_{n=0}^\infty g_n$ converges in $L^2(\mathbf{R})$ to some function $g \in L^2(\mathbf{R})$, and calculate $\|g\|_2$.

Comment: Difficulty rating: 1.

Solution: According to the Plancherel Theorem, the Fourier transform $f \mapsto \widehat{f}$ is an isometric linear bijection from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$. It is therefore equivalent to prove that $\sum_{n=0}^\infty \widehat{g_n}$ converges in $L^2(\mathbf{R})$ to some function $h \in L^2(\mathbf{R})$, and to calculate $\|h\|_2 = \|\widehat{g}\|_2$.

We have $\|\widehat{g_n}\|_2 = \|g_n\|_2 = \frac{1}{2^n}$. The series $\sum_{n=0}^\infty \widehat{g_n}$ therefore converges absolutely, so, by completeness, the series converges. Moreover, since $\widehat{g_m} \cdot \widehat{g_n} = 0$ almost everywhere when $m \neq n$, the elements $\widehat{g_n}$ are orthogonal. Therefore

$$\left\| \sum_{n=0}^\infty \widehat{g_n} \right\|^2 = \sum_{n=0}^\infty \|\widehat{g_n}\|^2 = \sum_{n=0}^\infty \frac{1}{4^n} = \frac{4}{3}.$$

So $\|h\|_2 = \|\widehat{g}\|_2 = 2/\sqrt{3}$. ∎

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7. Let $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function. Suppose that $f|_{\mathbf{R}}$ is periodic. Prove that f is periodic, that is, that there is $a \in \mathbf{C} \setminus \{0\}$ such that $f(z+a) = f(z)$ for all $z \in \mathbf{C}$.

Comment: Difficulty rating: 1.

Solution: By assumption, there is $a \in \mathbf{R} \setminus \{0\}$ such that $f(z+a) = f(z)$ for all $z \in \mathbf{R}$. Now $g(z) = f(z+a) - f(z)$ is an entire function which is zero on $\mathbf{R}$. Since $\mathbf{R}$ has a cluster point in the interior of the domain, it follows that $g(z) = 0$ for all $z \in \mathbf{C}$. This means that $f(z+a) = f(z)$ for all $z \in \mathbf{C}$. ∎

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8. Let f be the meromorphic function on $\mathbf{C}$ given by

$$f(z) = \frac{e^z}{z} + \frac{17}{z+4}.$$

Evaluate $\int_\gamma f(z)\, dz$, where $\gamma\colon [0, 2\pi] \to \mathbf{C}$ is given by $\gamma(t) = 3e^{it}$.

Comment: Difficulty rating: 1.

Solution: Clearly γ is a closed curve. The Residue Theorem tells us that

$$\frac{1}{2\pi i}\int_\gamma f(z)\, dz$$

is the sum of the residues of f at its poles, each multiplied by the winding number of γ about the corresponding pole. The poles of f are only at 0 and 4. A direct computation shows that the winding number of γ about 0 is 1. (This is geometrically obvious, and I don't expect the calculation to be carried out. However, something must be said.) The point 4 is in the unbounded component of $\mathbf{C} \setminus \gamma([0, 2\pi])$, so the winding number of γ about 4 is 0.

We need only find the residue at 0. We can write

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} + \frac{17}{z+4},$$

where the series converges for all z (to $(e^z - 1)/z$ for $z \neq 0$). Since all terms except the first are holomorphic on the open ball $B_4(0)$, the residue of f at 0 is 1.

It follows that

$$\int_\gamma f(z)\, dz = 2\pi i.$$

This completes the calculation. ∎

9. Let $f\colon \mathbf{C} \setminus \mathbf{Z} \to \mathbf{C}$ be a bounded holomorphic function. Prove that f is constant.

Comment: Difficulty rating: 1.

This is very close to part of one of the homework problems in Rudin (assigned Spring 2004): If f and g are entire functions, and $|f| \leq |g|$ everywhere, what can you say about the relation between f and g?

Solution: For each $n \in \mathbf{Z}$, the function f has an isolated singularity at n. Since f is bounded on the punctured ball $B_1(n) \setminus \{n\}$, it follows that this singularity is removable. Therefore there is an entire function g such that $g|_{\mathbf{C}\setminus\mathbf{Z}} = f$. By continuity, g is also bounded. Therefore Liouville's Theorem implies that g is constant. So f is constant. ∎