Analysis Qualifying Examination

Wniter 2003

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let A be the set of all continuous functions defined in the closed unit disk which are also analytic on U, where U is the unit disk. We know A is a linear space. Define

$$\|f\| = \sup\{|f(z)| : |z| \le 1\}.$$

Prove that A is closed in $C(\overline{U})$.

2. Suppose that f is an entire function such that its range is not dense in $\mathbf{C}$. Prove that f is a constant.

3. Let $p(z)$ be a polynomial of degree at least 2. Prove that the sum of residues of $1/p(z)$ at all zeros of p must be zero.

4. Suppose that $g_n \in L^1([0,\pi])$ such that

$$\lim_{n\to\infty} \int_{[a,b]} g_n dx = 0$$

for all $[a,b] \subset [0,\pi]$. Prove that, for any $f \in L^1([0,\pi])$,

$$\lim_{n\to\infty} \int_{[0,\pi]} f g_n dx = 0.$$

5. Let $(X, \mathcal{S}, \mu)$ be a measurable space with $\mu(X) < \infty$. Suppose that f is a complex measurable function. Prove

$$\lim_{n\to\infty} \mu(\{x \in X : |f(x)| > n\} = 0.$$

6. Suppose that $f(x)$ is non-decreasing in $[0,1]$. Show that

$$\int_{[0,1]} f'(x)dx \le f(b) - f(a).$$

7. Let X be a normed space and $f : X \to \mathbf{C}$ be a linear functional. Show that f is continuous if and only if

$$\{x \in X : f(x) = 0\}$$

is closed.

8. Let $f \in L^p([0,1])$. Define $T : L^p([0,1]) \to L^1([0,1])$ by $T(g) = fg$ for $g \in L^q([0,1])$, where $1/p + 1/q = 1$. Prove that T is bounded and find its norm.

9. Let $(X, \mathcal{B})$ be a measurable space and μ_n be a sequence of complex measures on X. Suppose that there is a complex measure μ (note that $|\mu_n|(X) < \infty$ and $|\mu|(X) < \infty$) such that

$$\lim_{n\to\infty} \int_X f\mu_n = \int_X f d\mu$$

for all bounded Borel measurable functions on X. Show that there is $M > 0$ such that

$$|\mu_n|(X) \leq M \text{ for all } n.$$