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Analysis Qualifying Exam Winter 2002
(with solutions)

1. Let $f : X \to [0,\infty]$ be a measurable function on a measure space (X,μ). with $\mu(X) < \infty$. Prove that $f \in L^1(X)$ if and only if

$$\sum_{n\geq 0} 2^n \mu\{x \in X | f(x) \geq 2^n\} < \infty.$$

Solution:
Let $F_n = \{x \in X | f(x) \geq 2^k\}$. Consider disjoint sets

$$E_k = \{x \in X | 2^{k-1} \leq f(x) < 2^k\} \text{ if } k \geq 1 \quad \text{and} \quad E_0 = \{x \in X | f(x) < 1\}.$$

Then, since

$$F_n = \coprod_{k>n} E_k \quad \text{and} \quad X = \coprod_{k\geq 0} E_k,$$

we have

$$\sum_{k\geq 0} 2^{k-1} \chi_{E_k} \leq f \leq \sum_{k\geq 0} 2^k \chi_{E_k}$$

or $g/2 \leq f \leq g$ where $g = \sum_{k\geq 0} 2^k \chi_{E_k}$. The sets E_k are measurable because f is measurable, hence g is measurable. Since both f and g are non-negative, f is in L^1 if and only if g is. On the other hand,

$$\sum_{n\geq 0} 2^n \mu\{x \in X | f(x) \geq 2^n\} = \sum_{n\geq 0} 2^n \mu(F_n) = \sum_{n\geq 0} 2^n \sum_{k\geq n+1} \mu(E_k)$$

$$= \sum_{k\geq 1} \mu(E_k) \sum_{n=0}^{k-1} 2^n = \sum_{k\geq 1} (2^k - 1)\mu(E_k) = \int_X g d\mu - \mu(E_0) - \mu(X).$$

∎

2. Find each limit and justify your answers. Quote all the theorems you are using to manipulate the limits and integrals and verify their applicability.

(a) $\displaystyle\lim_{n\to\infty} \int_{\frac{1}{n}}^{n} e^{\frac{1}{n}\sqrt{x}-x} dx.$

(b) $\displaystyle\int_0^{\infty} \int_y^{\infty} e^{-x^2} dxdy.$

Solution:
(a) First, notice that the sequence of functions $f_n(x) = e^{\frac{1}{n}\sqrt{x}-x}\chi_{[\frac{1}{n},n]}$ on $[0,\infty)$ converges pontwise to $f(x) = \lim_{n\to\infty} f_n(x) = e^{-x}$. Since each f_n is piece-wise continuous, hence measurable, and $0 \leq f_n(x) \leq f(x)$ for all $x \in [0,\infty)$ and $f \in L^1$, the Dominated Convergence Theorem applies, giving

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{n} e^{\frac{1}{n}\sqrt{x}-x} dx = \lim_{n\to\infty} \int_0^{\infty} f_n(x)\, dx = \int_0^{\infty} f(x)\, dx = \int_0^{\infty} e^{-x}\, dx = 1.$$

∎

(b) The function $f(x,y) = e^{-x^2}$ is continuous on $\mathbf{R}^2$, hence measurable. Further let

$$E = \{(x,y) \in \mathbf{R}^2 : 0 \leq y \leq x\}.$$

Then E is a closed subset of $\mathbf{R}^2$, hence measurable. It follows that $g = \chi_E f$ is measurable on $\mathbf{R}^2$. The integral to be evaluated can be written as

$$\int_{\mathbf{R}} \int_{\mathbf{R}} g(x,y)\, dxdy.$$

Since g is nonnegative and Lebesgue measure on $\mathbf{R}$ is σ-finite, we can apply the version of Fubini's Theorem for positive measurable functions:

$$\begin{aligned} \int_{\mathbf{R}} \int_{\mathbf{R}} g(x,y)\, dxdy &= \int_{\mathbf{R}} \int_{\mathbf{R}} g(x,y)\, dydx = \int_0^\infty \int_0^x \exp(-x^2)\, dydx \\ &= \int_0^\infty x \exp(-x^2)\, dx = \frac{1}{2} \end{aligned}$$

(the last integral is evaluated using an obvious substitution). ∎

3. (a) State the Radon-Nikodym theorem.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space. Denote by $M(X)$ the space of all complex measures defined on the σ-algebra $\mathcal{M}$.

Let $E \subset M(X)$ be the set of all measures in $M(X)$ which are absolutely continuous with respect to μ. Prove that E is a closed subspace of $M(X)$.

Solution:

(a) Let μ be a positive σ-finite measure on a σ-algebra $\mathcal{M}$ om a set X, and let ν be a complex measure on X. Then there exists a unique pair of complex measures ν_a and ν_s on $\mathcal{M}$ such that

$$\nu = \nu_a + \nu_s, \quad \nu_a << \mu, \quad \nu_s \perp \mu.$$

(b) The proof that E is a vector subspace is easy, and is omitted. Let us show that E is closed. Let (ν_n) be a sequence in E and suppose $||\nu_n - \nu|| \to 0$. For $B \in \mathcal{M}$, we have

$$|\nu_n(B) - \nu(B)| \leq |\nu_n(B) - \nu(B)| + |\nu_n(X \setminus B) - \nu(X \setminus B)| \leq ||\nu_n - \nu||.$$

Therefore $\nu_n(B) \to \nu(B)$ for all $B \in \mathcal{M}$.

Let $N \in \mathcal{M}$ satisfy $\mu(N) = 0$. Then

$$\nu(N) = \lim_{n\to\infty} \nu_n(N) = \lim_{n\to\infty} 0 = 0.$$

This shows that ν is absolutely continuous with respect to μ, as desired.

∎

4. Let p, q, r be real numbers in $[1, \infty)$ satisfying $1/r = 1/p + 1/q$. Prove that for functions $f \in L^p$ and $g \in L^q$ their product fg belongs to L^r and that the multiplication operator

$$M_f : L^q \to L^r, \quad M_f(g) = fg,$$

is a bounded operator with norm $||M_f|| = ||f||_p$.

Solution:

Since $\frac{r}{p} + \frac{r}{q} = 1$ we have by Hölder's inequality

$$\begin{aligned} \|fg\|_r &= \left(\int |fg|^r\right)^{1/r} = \left(\int |f^r||g^r|\right)^{1/r} \leq (\||f^r\|_{p/r}\|g^r\|_{q/r}\|)^{1/r} \\ &= \left(\left(\int |f^r|^{p/r}\right)^{r/p}\left(\int |g^r|^{q/r}\right)^{r/q}\right)^{1/r} = \|f\|_p\|g\|_q. \end{aligned}$$

This implies that $fg \in L^r$. Also we have $\|M_f(g)\|_r \leq \|f\|_p\|g\|_q$ which means that M_f is bounded and $\|M_f\| \leq \|f\|_p$.

To prove the lower bound for $\|M_f\|$, notice that $h = f^{p/q}$ is in L^q with

$$\|h\|_q = \left(\int |f^{p/q}|^q\right)^{1/q} = \left(\int |f|^p\right)^{1/q} = \|f\|_p^{p/q}$$

and

$$\|M_f(h)\|_r = \|f^{1+p/q}\|_r = \left(\int |f|^{r(1+p/q)}\right)^{1/r} = \left(\int |f|^p\right)^{1/r} = \|f\|_p^{p/r}.$$

This gives the desired lower bound for $\|M_f\|$:

$$\|M_f\| \geq \|M_f(h)\|_r/\|h\|_q = \|f\|_p^{p/r-p/q} = \|f\|_p,$$

since $p(\frac{1}{r} - \frac{1}{q}) = 1$. ∎

5. Let V be a Banach space, and let (ϕ_n) be a sequence of continuous linear functionals on V, such that the limit

$$\phi(x) = \lim_{n\to\infty} \phi_n(x)$$

exists for all $x \in V$.

Prove that ϕ is a continuous linear functional on V.

Solution:

Linearity of ϕ is clear:

$$\phi(ax + by) = \lim_{n\to\infty} \phi_n(ax + by) = \lim_{n\to\infty} (a\phi_n(x) + b\phi_n(y)) = a\phi(x) + b\phi(y).$$

For linear functionals on a normed space continuity is equivalent to boundedness, therefore ϕ_n is bounded.

To prove that ϕ is bounded we use the Banach-Steinhaus Theorem.

Let $x \in V$. Since $\phi_n(\xi) \to \phi(\xi)$, the set $\{\phi_n(x) | n \in \mathbf{N}\}$ is bounded. Since V is complete and the ϕ_n are bounded, the Banach-Steinhaus Theorem implies that $M = \sup_{n\in\mathbf{N}} \|\phi_n\| < \infty$. Therefore, for every $x \in V$,

$$|\phi(x)| = \lim_{n\to\infty} |\phi_n(x)| \leq M\|x\|.$$

This shows that $\|\phi\| \leq M$, so that ϕ is bounded. ∎

6. (a) Give the definition of a Hilbert space.

(b) Let F and H be Hilbert spaces, with scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. Define a scalar product on the vector space direct sum $F \oplus H$ by

$$\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle = \langle \xi_1, \xi_2 \rangle_1 + \langle \eta_1, \eta_2 \rangle_2.$$

Prove that this makes $F \oplus H$ a Hilbert space. (Be sure to prove completeness!)

Solution: (a) A Hilbert space is a complex vector space H equipped with a scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbf{C}$ satisfying the following properties:

(1) $\langle \eta, \xi \rangle = \overline{\langle \xi, \eta \rangle}$ for all ξ, $\eta \in H$.

(2) $\langle \xi_1 + \xi_2, \eta \rangle = \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta \rangle$ for all ξ_1, ξ_2, $\eta \in H$.

(3) $\langle \alpha\xi, \eta \rangle = \alpha \langle \xi, \eta \rangle$ for all ξ, $\eta \in H$ and $\alpha \in \mathbf{C}$.

(4) $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in H$.

(5) $\langle \xi, \xi \rangle = 0$ only if $\xi = 0$.

(6) H is complete in the norm defined by $||\xi|| = \langle \xi, \xi \rangle^{1/2}$.

(b) The verification of the axioms (1) through (4) above for the scalar product on $F \oplus H$ is completely routine, and is omitted from the solutions. For (5), assume that

$$\langle (\xi, \eta), (\xi, \eta) \rangle = 0.$$

This means that

$$\langle \xi, \xi \rangle_1 + \langle \eta, \eta \rangle_2 = 0.$$

Since

$$\langle \xi, \xi \rangle_1 \geq 0 \quad \text{and} \quad \langle \eta, \eta \rangle_2 \geq 0,$$

this implies that

$$\langle \xi, \xi \rangle_1 = \langle \eta, \eta \rangle_2 = 0.$$

So $(\xi, \eta) = 0$.

It remains to prove (6) (completeness).

Let (ξ_1, η_1), (ξ_2, η_2), ... be a Cauchy sequence in $F \oplus H$. We observe the inequality, for any $\xi \in F$ and $\eta \in H$,

$$||(\xi, \eta)|| = \left(\langle \xi, \xi \rangle_1 + \langle \eta, \eta \rangle_2 \right)^{1/2} \geq \max(||\xi||, ||\eta||).$$

It follows from this inequality that the sequences $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ are Cauchy in F and H respectively. Therefore they have limits $\xi \in F$ and $\eta \in H$. We prove that $||(\xi, \eta) - (\xi_n, \eta_n)|| \to 0$.

Using the formula for the norm and the scalar product, we find

$$\begin{aligned} ||(\xi, \eta) - (\xi_n, \eta_n)||^2 &= ||(\xi - \xi_n, \eta - \eta_n)||^2 \\ &= \langle \xi - \xi_n, \xi - \xi_n \rangle_1 + \langle \eta - \eta_n, \eta - \eta_n \rangle_2 \\ &= ||\xi - \xi_n||^2 + ||\eta - \eta_n||^2. \end{aligned}$$

Since

$$||\xi - \xi_n|| \to 0 \quad \text{and} \quad ||\eta - \eta_n|| \to 0,$$

it follows that $||(\xi, \eta) - (\xi_n, \eta_n)|| \to 0$. ∎

7. Describe all entire functions $f : \mathbf{C} \mapsto \mathbf{C}$ for which there exist constans a and b such that

$$|f(z)| \leq a|z|^{5/2} + b \qquad \forall z \in \mathbf{C}.$$

Solution:

By Cauchy's integral formula for derivatives,

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}} dz, \qquad R > 0,$$

we get the estimate

$$\begin{aligned} |f^{(k)}(0)| &\leq \tfrac{k!}{2\pi} \int_{|z|=R} \tfrac{|f(z)|}{|z|^{k+1}} |dz| \\ &\leq \tfrac{k!}{2\pi R^{k+1}} \int_{|z|=R} (|a|z|^{5/2} + b)|dz| \\ &= \tfrac{k!(aR^{5/2}+b)}{R^k} \end{aligned}$$

If $k \geq 3$ then the right hand side of the above estimate goes to zero. By Taylor expansion, $f(z) = f(0) + f'(0)z + f''(0)z^2/2$. So that f is a polynomial of degree at most 2.

∎

8. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx.$$

Solution:

Consider the function $f(z) = (e^{iz})/(z^4 + 1)$ and integral of f over the contour Γ_R $R > 0$, which consists of 2 parts: the segments of real line from $-R$ to R and the upper half circle $\Gamma_R^+(t) = Re^{it}$, $0 \leq t \leq \pi$. We have

$$\Big| \int_{\Gamma_R^+} \frac{e^{iz}}{z^4 + 1} dz \Big| \leq \int_{\Gamma_R^+} \frac{|e^{iz}|}{R^4 + 1} |dz| \leq \frac{\pi R}{R^4 + 1}$$

which goes to zero as $R \to \infty$. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx &= \lim_{R \to \infty} \Re\Big(\int_{\Gamma_R} \frac{e^{iz}}{z^4 + 1} dz \Big) \\ &= \Re \left(2\pi i \mathrm{Res}\Big(\frac{e^{iz}}{z^4 + 1}, e^{\pi i/4} \Big) + 2\pi i \mathrm{Res}\Big(\frac{e^{iz}}{z^4 + 1}, e^{3\pi i/4} \Big) \right) \\ &= -2\pi \Im \Big(\frac{e^{iz}}{4z^3} \Big|_{z=e^{i\pi/4}} + \frac{e^{iz}}{4z^3} \Big|_{z=e^{3\pi i/4}} \Big) \\ &= \frac{\pi}{\sqrt{2}} e^{-1/\sqrt{2}} \Big(\cos \frac{\sqrt{2}}{2} + \sin \frac{\sqrt{2}}{2} \Big) \end{aligned}$$

∎

9. Let f be a continuous complex valued function on the interval $[0,1]$, and define function function F by

$$F(z) = \int_0^1 f(t)e^{tz}dt, \qquad z \in \mathbf{C}.$$

Prove that F is analytic in the entire complex plane.

Solution:

Solution 1 The function $f(t)e^{tz}$ is a continuous function of the variable $(t,z) \in [0,1] \times \mathbf{C}$. If $R \subset \mathbf{C}$ is a rectangle, then by Fubini's theorem and the analyticity of e^{tz} with respect to z,

$$\int_R F(z)dz = \int_R \int_0^1 f(t)e^{tz}dtdz = \int_0^1 \int_R f(t)e^{tz}dzdt = 0.$$

By Morera's theorem, F is analytic.

Solution 2 Let $z \in \mathbf{C}$. Since $f(t)$ is bounded, the series

$$f(t)e^{tz} = f(t)\sum_{n=0}^{\infty} \frac{t^n z^n}{n!}$$

converges uniformly for $t \in [0,1]$, so that we can change the order of summation and integration to get

$$F(z) = \int_0^1 f(t)e^{zt}dt = \sum_{n=0}^{\infty} \int_0^1 f(t)t^n dt \frac{z^n}{n!}.$$

Since

$$\left|\frac{1}{n!}\int_0^1 f(t)t^n dt\right| \le \frac{1}{n!}\int_0^1 |f(t)|dt,$$

the radius of convergence of the series of F is ∞.

∎