Analysis Qualifying Examination
Winter 2001

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let X be a connected subset of a metric space and $f : X \to X$ be a continuous map. Show that $f(X)$ is connected.

2. Let $f_n \in L^2(X, \mu)$ such that $f_n \to 0$ in $L^2(X, \mu)$. Show that $f_n \to 0$ in measure.

3. Evaulate (Justify your computation)

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin(x/n)}{x\log(1+x)} dx.$$

4. Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$. Suppose that for any complex Borel measure μ with $\mu([0, 1]) = 1$

$$\left| \int_{[0,1]} f_n d\mu \right| \leq \pi.$$

Show that $\{||f_n||\}$ is bounded.

5. Let H be a Hilbert space and T, K be two bounded operators from H to H. Suppose that K is a compact operator (i.e., K maps every bounded subset of H to a pre-compact subset).

(a) Show that TK is also compact.

(b) Show that if H is infinite dimensional, then K is not invertible.

6. Let $f \in L^1([0, 1])$. Suppose that $f(x) = \int_0^x f(t)dt$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

7. Let D be the open unit disk and $A(D)$ be the set of continuous functions on the closed unit disk and holomorphic on the open unit disk. Given $A(D)$ is a subspace of $C(\bar{D})$. Show that $A(D)$ is closed.

8. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{2x^2}{x^4+1} dx.$$

9. Let f be an entire function such that $|f(z)| \leq |z|^{1/2}$. Show that f is a constant.