Analysis Qualifying Exam, Fall 2011

1. Let $C_0[0,1]$ be the space of all continuous functions f on $[0,1]$ such that $f(0)=0$, with the supremum $||\cdot||_\infty$ norm. Let

$$M = \{f \in C_0[0,1] : \int_0^1 f(t)dt = 1\}.$$

Prove that M is a closed convex subset of $C_0[0,1]$ which contains no element of minimal norm.

2. Prove that for any complex Lebesgue measurable function f on $\mathbb{R}$ and $\varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{C}$ such that the set

$$\{x \in \mathbb{R} : f(x) \neq g(x)\}$$

has Lebesgue measure $< \varepsilon$.

3. Let $L^1[0,1]$ be the space of all Lebesgue integrable functions on $[0,1]$ with the usual $||\cdot||_1$ norm. Let g be a Lebesgue measurable function on $[0,1]$. Prove that $||g||_\infty < \infty \iff$ for all $f \in L^1[0,1]$ we have

$$\int_0^1 |f(t)g(t)|dt < \infty.$$

4. Let μ be a **complex** Borel measure on $\mathbb{R}$. Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is a collection of Borel functions on $\mathbb{R}$ such that

$$\sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}} |f_k(x)| < \infty \qquad \text{and} \qquad f(x) = \lim_{k\to\infty} f_k(x) \text{ exists for all } x \in \mathbb{R}.$$

Prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k d\mu = \int_{\mathbb{R}} f d\mu.$$

5. Suppose that $f : [a,b] \to \mathbb{R}$ is AC (absolutely continuous). Prove that that there exist non-decreasing AC functions $f_1, f_2 : [a,b] \to \mathbb{R}$ such that $f = f_1 - f_2$.

6. Let c_0 consists of all sequences $x = (x_k)_{k\in\mathbb{N}}$ such that $x_k \to 0$ as $k \to \infty$ with the norm

$$||x||_\infty = \sup_{k\in\mathbb{N}} |x_k|.$$

Prove that the dual space $(c_0)^*$ can be identified with the space ℓ^1, which consists of all summable sequences on $\mathbb{N}$.

7. Let $f \in L^1(\mathbb{R})$ and $g = \chi_{[0,1]}$. Suppose that their convolution

$$f * g(x) = 0 \qquad \text{for a.e. } x \in \mathbb{R}.$$

Prove that $f = 0$ a.e.

8. Let f be a holomorphic function on the open disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $||f||_\infty = \sup_{z\in D} |f(z)| < 1$. Prove that f has exactly one fixed point $z_0 \in D$, i.e., $f(z_0) = z_0$.

9. Let X be the closure of the set $\{1/n + 1/m : n, m \in \mathbb{N}\}$. Suppose that f is a holomorphic function on $\mathbb{C} \setminus X$. Prove that if f is bounded on $\{z \in \mathbb{C} \setminus X : |z| < 3\}$, then f extends to an entire function.