Analysis Qualifying Exam (F2010)

1. Let f be a continuous function on $\mathbb{R}$. Suppose that $f = 0$ a.e.m.(with respect to the Lebesgue measure). Prove that $f(x) = 0$ for every $x \in \mathbb{R}$.

2. Suppose that $f_n \in L^2(\mathbb{R})$ and $f_n \to 0$ in $L^2(\mathbb{R})$. Suppose also that $|f_n(x)| \leq \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dm = 0.$$

3. Let $f \in L^1([0,1])$. Define $F(t) = \int_0^t f(x)dm$. Prove directly that F has bounded variation.

4. Let c be the set of convergent sequences. It is a linear space. Let $\xi = \{x_n\} \in c$. Define

$$\|\xi\| = \sup\{|x_n| : n = 1, 2, ...\}.$$

Prove directly that c is complete.

5. Let f_n and f be continuous on $[0,1]$. Suppose that, for any probability Borel measure μ,

$$\lim_{n\to\infty} \int_{[0,1]} f_n d\mu = \int_{[0,1]} f d\mu.$$

Prove that $\{f_n\}$ is a bounded sequence in $C([0,1])$.

6. Let H be a Hilbert space and let $H_0 \subset H$ be a finite dimensional subspace. Suppose that $T : H \to H_0$ is a bounded linear map. Prove that there exists a complex number λ such that $T(\xi) = \lambda\xi$ for some non-zero $\xi \in H$.

7. Prove the following theorem: Let Ω be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence of holomorphic functions on Ω. Suppose that f_n converges uniformly to f on every compact subset of Ω. Then, f is holomorphic on Ω and f'_n converges uniformly to f' on every compact subset of Ω.

8. Let f be an entire function which has no zero and let $R > 0$. Prove that there exists an integer $N > 0$ such that $P_N(z) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} z^k$ has no zero in $|z| \leq R$.

9. Compute

$$\int_0^\infty \frac{dx}{1+x^6}.$$