Analysis Qualifying Examination

F 2009

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let m be the Lebesgue measure and let E be a bounded subset of the real line. Show that E is Lebesgue measurable if and only if there are an F_σ set $F \subset E$ and a G_δ set $G \supset E$ such that

$$m(G \setminus E) = 0.$$

2. Let $f_n(x) = \frac{nx}{1+n^2x}$. Find the limit

$$\lim_{n\to\infty} \int_{[0,1]} f_n dm,$$

where m is the Lebesgue measure (Justify your answer!).

3. Suppose that $\{f_n\}$ is a sequence in $L^2(\mathbf{R})$ which converges to zero almost everywhere in $\mathbf{R}$. Does f_n converge to zero in measure? Why? What if f_n converges to zero in $L^2(\mathbf{R})$? Why?

4. Let $\{f_n\}$ be a sequence of functions which have bounded variation on $[0, 1]$. Suppose that $V_0^1(f_n) \leq 1$ for all n and $\{f_n\}$ converges to a function f uniformly on $[0, 1]$. Prove that f has bounded variation. Give an estimate of $V_0^1(f)$.

5. Let c_0 be the complex space of all sequences which converge to zero. Define $\|\{x(k)\}\| = \sup_k |x(k)|$. Verify directly that c_0 with this norm is a Banach space (do not verify it is a norm or verify c_0 is a linear space).

6. Let X be a Banach space and let $\{x_n\}$ be a sequence of elements in X. Suppose that, for each bounded linear functional f, $\{f(x_n)\}$ converges. Show that $\{\|x_n\|\}$ is a bounded sequence.

7. Let f be a Borel function defined and bounded on the closed unit disk which is holomorphic in the open unit disk. Suppose also that $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ almost everywhere with respect to the Lebesgue measure on the unit circle. Evaluate

$$\int_C \frac{f(z)}{z} dz,$$

where C is the positively oriented unit circle.

8. Let f be a holomorphic function on the open unit disk. Suppose that

$$\lim_{|z|\to 1} f(z) = 1.$$

Prove that $f(z) = 1$ for all z in the open unit disk.

9. Let f be holomorphic in $\{z : 0 < |z| < 1\}$ with simple pole at the origin. Find

$$\lim_{r\to 0} \int_{\gamma_r} f(z)dz,$$

where γ_r is the upper semi-circle with the radius r and the center 0.