Qualifying Exam Analysis Fall 2008
Solutions

Problem 1. Let μ be a positive measure on X. Recall that a sequence $\{f_n\}$ of complex measurable functions on X is said to **converge in measure** to the measurable function f_∞ if for any $\epsilon > 0$ there is a n_0 such that

$$\mu(\{x, |f_n(x) - f_\infty(x)| > \epsilon\} < \epsilon \quad \text{for } n > n_0.$$

Let $\{g_n\}$ and $\{h_n\}$ be two sequences of measurable functions on X which converge in measure to the complex-valued measurable functions g_∞ and h_∞ respectively. Assume $\mu(X) < \infty$, prove that $\{g_n h_n\}$ converge in measure to the measurable functions $g_\infty h_\infty$.

Solution to Problem 1. Since $\mu(X) < \infty$, given any $\epsilon \in (0,1)$, there is a constant $A > 6$ such that

$$\mu(\{x, |g_\infty(x)| > A) < \frac{\epsilon}{6} \quad \text{and} \quad \mu(\{x, |h_\infty(x)| > A) < \frac{\epsilon}{6}. \tag{1}$$

By definition of convergence in measure there is a n_0 such that for $n > n_0$

$$\mu(\{x, |g_n(x) - g_\infty(x)| > \frac{\epsilon}{2(A+1)}\} < \frac{\epsilon}{6},$$
$$\mu(\{x, |h_n(x) - h_\infty(x)| > \frac{\epsilon}{2(A+1)}\} < \frac{\epsilon}{6}.$$

When $n > n_0$ for x satisfying $|g_\infty(x)| \leq A$ we have $|g_n(x)| \leq A + 1$. From

$$|(g_n h_n)(x) - (g_\infty h_\infty)(x)| = |g_n(x)(h_n(x) - h_\infty(x)) + h_\infty(x)(g_n(x) - g_\infty(x))|$$

we have

$$\begin{aligned}
&\{x, |(g_n h_n)(x) - (g_\infty h_\infty)(x)| > \epsilon, |g_\infty(x)| \leq A, |h_\infty(x)| \leq A\} \\
&\subset \{x, |g_n(x)(h_n(x) - h_\infty(x))| > \frac{\epsilon}{2}, |g_\infty(x)| \leq A, |h_\infty(x)| \leq A\} \\
&\cup \{x, |h_\infty(x)(g_n(x) - g_\infty(x))| > \frac{\epsilon}{2}, |g_\infty(x)| \leq A, |h_\infty(x)| \leq A\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mu(\{x, |g_n(x)(h_n(x) - h_\infty(x))| > \frac{\epsilon}{2}, |g_\infty(x)| \leq A, |h_\infty(x)| \leq A\}) \\
&\leq \mu(\{x, |h_n(x) - h_\infty(x)| > \frac{\epsilon}{2(A+1)}\}) \leq \frac{\epsilon}{6}.
\end{aligned}$$

and

$$\mu(\{x, |h_\infty(x)(g_n(x) - g_\infty(x))| > \frac{\epsilon}{2}, |g_\infty(x)| \le A, |h_\infty(x)| \le A\})$$
$$\le \mu(\{x, |g_n(x) - g_\infty(x)| > \frac{\epsilon}{2A+1)}\}) \le \frac{\epsilon}{6}.$$

We have proved for $n > n_0$

$$\mu(\{x, |(g_n h_n)(x) - (g_\infty h_\infty)(x)| > \epsilon, |g_\infty(x)| \le A, |h_\infty(x)| \le A\}) \le \frac{\epsilon}{3} \qquad (2)$$

Note that

$$\{x, |(g_n h_n)(x) - (g_\infty h_\infty)(x)| > \epsilon\} \subset \{x, |g_\infty(x)| > A\} \cup \{x, |h_\infty(x)| > A\}$$
$$\cup \{x, |(g_n h_n)(x) - (g_\infty h_\infty)(x)| > \epsilon, |g_{\inf ty}(x)| \le A, |h_\infty(x)| \le A\},$$

it follows from (1) and (2) we have for $n > n_0$

$$\mu(\{x, |(g_n h_n)(x) - (g_\infty h_\infty)(x)| > \epsilon\}) < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3} < \epsilon.$$

Hence $g_n h_n$ converges to $g_\infty h_\infty$ in measure.

Problem 2. Let $\{f_n\}$ be a sequence of complex Borel functions on $(0,1]$. Assume $|f_n(x)| \le 2009$ for all $x \in (0,1]$ and $\lim_{n\to\infty} f_n(x) = 2008$ almost everywhere with respect to Lebesgue measure dx on $(0,1]$. Find with justification the following limit

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{2x^{1/2}\cos(x/n)} dx.$$

Solution to Problem 2. Let $g(x) = \frac{2009}{2x^{1/2}\cos 1}$. It is clear from assumption that $\left|\frac{f_n(x)}{2x^{1/2}\cos(x/n)}\right| \le g(x)$. Note

$$\int_0^1 g(x)dx < \infty,$$

it follows from Lebesgue dominant convergence theorem that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{2x^{1/2}\cos(x/n)} dx = \int_0^1 \lim_{n\to\infty} \frac{f_n(x)}{2x^{1/2}\cos(x/n)} dx$$
$$= \int_0^1 \frac{2008}{2x^{1/2}} dx = 2008.$$

Problem 3. Let $E \subset \mathbb{R}^1$ be a set with Lebesgue measure $m(E) > 0$. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. Assume function

$$F(x) = \int_E f(x-t)dt \in L^1(\mathbb{R}).$$

Show that $f \in L^1(\mathbb{R})$.

Solution to Problem 3. First we show that $f(x-t)$ as a function of $(x,t) \in \mathbb{R}^2$ is measurable. Let $G(x,t) = x - t$. Then $f(x-t) = f \circ G(x,t)$. Since f is Borel measurable and G is continuous (hence measurable), by Theorem 1.12(d) on p.13 of [Rudin] we conclude that $f(x-t)$ is measurable on $\mathbb{R}^2$.

Since $F(x) \in L^1(\mathbb{R})$, by Fubini theorem we have Note that

$$\int_E \int_{\mathbb{R}} f(x-t)dxdt = \int_E \|f\|_{L^1(\mathbb{R})}dt = \|f\|_{L^1(\mathbb{R})} \cdot m(E) < \infty.$$

Hence it follows from $m(E) > 0$ that $\|f\|_{L^1(\mathbb{R})} < \infty$.

Problem 4. Let $(X, \|\cdot\|)$ be a normed space and let $x_1, \cdots, x_{2010}$ be a linearly independent subset. Show that there is a bounded linear function $F : X \to \mathbb{R}$ such that

$$F(x_i) = 100^i \qquad \text{for } i = 1, \cdots, 2010.$$

Solution to Problem 4. Let E be the linear span of $\{x_1, \cdots, x_{2010}\}$ in X. Since $x_1, \cdots, x_{2010}$ are linearly independent, hence we can define a linear functional f on E by

$$f(c_1x_1 + \cdots + c_{2010}x_{2010}) = c_1 \cdot 100^1 + \cdots + c_{2010} \cdot 100^{2010}.$$

To extend f to the required functional F by Hahn-Banach theorem we need to show that f is bounded on $(E, \|\cdot\|)$, i.e., there is a M such that for all $x_1, \cdots, x_{2010}$

$$|c_1 \cdot 100^1 + \cdots + c_{2010} \cdot 100^{2010}| \le M \cdot \|c_1x_1 + \cdots + c_{2010}x_{2010}\|.$$

Since

$$|c_1 \cdot 100^1 + \cdots + c_{2010} \cdot 100^{2010}| \le 100^{2010}(|c_1| + \cdots + |c_{2010}|)$$

and

$$\|c_1x_1 + \cdots + c_{2010}x_{2010}\|_1 = |c_1| + \cdots + |c_{2010}|$$

is a norm on E. The required statement follows from that any norm on finite dimensional space are equivalent, i.e., there is a M_1 such that

$$|c_1| + \cdots + |c_{2010}| \le M_1 \cdot \|c_1x_1 + \cdots + c_{2010}x_{2010}\|$$

Problem 5. Recall that l^∞ is the set of bounded sequences of real numbers. Define $c = \{\{x_n\} \in l^\infty, \lim_{n\to\infty} x_n \text{ exists}\}$ with the induced supremum norm. Show

(i) For $\{\eta_n\} \in l^1$ and $\alpha \in \mathbb{R}$, functional

$$f(\{x_n\}) = \sum_{n=1}^{\infty} \eta_n x_n + \alpha \cdot \lim_{m\to\infty} x_m$$

is a bounded linear functional (i.e., in $(c)^*$).

(ii) The norm of the functional f is given by

$$\|f\| = \|\{\eta_n\}\|_{l^1} + |\alpha|.$$

Solution to Problem 5. (i) It is straight forward to check that f is a linear functional. To see that f is bounded, first note that

$$|\lim_{m\to\infty} x_m| \leq \sup_n\{|x_n|\}.$$

Hence we have

$$|f(x)| \leq (\sum_{n=1}^{\infty} |\eta_n|) \cdot \sup_n\{|x_n|\} + |\alpha| \cdot \sup_n\{|x_n|\} \leq (\|\{\eta_n\}\|_{l^1} + |\alpha|) \sup_n\{|x_n|\}.$$

(ii) We have proved that

$$\|f\| \leq \|\{\eta_n\}\|_{l^1} + |\alpha|.$$

To see the other direction, given any $\epsilon > 0$, we choose a n_0 large enough such that

$$\|\{\eta_n\}\|_{l^1} - \epsilon \leq \sum_{n=1}^{n_0} |\eta_n|.$$

In particular we have

$$\sum_{n=n_0+1}^{\infty} |\eta_n| \leq \epsilon.$$

Define $x = (sign(\eta_1), \cdots, sign(\eta_{n_0}), sign(\alpha), sign(\alpha), \cdots)$, then $\|x| = 1$ except the trivial case $\{\eta_n\} = 0$ and $\alpha = 0$. Hence we have

$$\|f\| \geq f(x) \geq \sum_{n=1}^{n_0} |\eta_n| - \epsilon + |\alpha| \geq \sum_{n=1}^{\infty} |\eta_n| - 2\epsilon + |\alpha|.$$

Problem 6. Let H_2 be the set of holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on unit disc $\{z \in \mathbb{C}, |z| < 1\}$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Prove

(i) H_2 with inner product

$$(f, g) = \lim_{r\to 1_-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{\sqrt{-1}\theta})\overline{g(re^{\sqrt{-1}\theta})} d\theta,$$

is a complex Hilbert space.

(ii) For any complete orthonormal basis $\{e_n(z)\}_{n=0}^{\infty}$, we have

$$\sum_{n=0}^{\infty} e_n(z)\overline{e_n(t)} = \frac{1}{1-z\bar{t}} \quad \text{for any } |z|<1,\ |t|<1.$$

Solution to Problem 6. Consider map

$$\Phi : H_2 \to l^2 \quad \text{with} \quad \Phi(f) = (a_0, a_1, \cdots).$$

Φ is a linear map. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$. It is easy to compute (students need to do the calculation)

$$(f,g) = \sum_{n=0}^{\infty} a_n \overline{b_n} = (\Phi(f), \Phi(g)).$$

Hence Φ is an isometry. Since l^2 is a Hilbert space, so it H_2.

(ii) Note $\{z_n\}_{n=0}^{\infty}$ is an orthonormal basis of H_2. Suppose $e_n(z) = \sum_{m=0}^{\infty} \alpha_{nm} z^m$, then we have $(e_n, e_m) = \sum_{k=0}^{\infty} \alpha_{nk}\overline{\alpha_{mk}} = \delta_{nm}$. It follows that $\sum_{k=0}^{\infty} \alpha_{kn}\overline{\alpha_{km}} = \delta_{nm}$ (taking product with a_{lm} to verify). We compute

$$\sum_{n=0}^{\infty} e_n(z)\overline{e_n(t)} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha_{nk} z^k \overline{\alpha_{nl}} \bar{t}^l =$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \delta_{kl} z^k \bar{t}^l = \sum_{k=0}^{\infty} z^k \bar{t}^k = \frac{1}{1-z\bar{t}}.$$

Problem 7. Evaluate

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx.$$

Solution to Problem 7. Consider the counterclockwise loop $[-R,R] \cup \{Re^{\sqrt{-1}\theta}, \theta \in [0,\pi]\}$. Since $\lim_{R\to\infty} \int_{\{Re^{\sqrt{-1}\theta},\theta\in[0,\pi]\}} \frac{z}{1+z^4} dz = 0$, we have

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = \int_{[-R,R]\cup\{Re^{\sqrt{-1}\theta},\theta\in[0,\pi]\}} \frac{z}{1+z^4} dz.$$

The later integration can be computed using residue. Note that $e^{\sqrt{-1}\frac{\pi}{4}}$ and $e^{\sqrt{-1}\frac{3\pi}{4}}$ are two simple poles. We have

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = 2\pi\sqrt{-1}\left(\frac{z}{4z^3}\Big|_{z=e^{\sqrt{-1}\frac{\pi}{4}}} + \frac{z}{4z^3}\Big|_{z=e^{\sqrt{-1}\frac{3\pi}{4}}} \right)$$

$$= \pi.$$

Problem 8. Let f be a holomorphic function on $\{z \in \mathbb{C}, |z| < 2\}$. Assume $f(z) \neq 0$ for any $|z| = 1$. Show

$$\operatorname{Re}\left(\int_{|z|=1} \frac{f'}{f}\, dz\right) = 0.$$

Solution to Problem 8. Since f has finite number of poles in the unit disc. Let z_i be pole of order r_i, $i = 1, \cdots, n$ be the list of poles. In a neighborhood of z_i we can write $f(z) = (z - z_i)^{r_i} g_i(z)$ where $g(z_i) \neq 0$, we can compute the residue of $\frac{f'}{f}$ at z_i which is a simple pole by

$$\lim_{z \to z_i} \frac{f'}{f}(z - z_i) = \lim_{z \to z_i} \frac{r_i (z - z_i)^{r_i - 1} g_i(z)}{(z - z_i)^{r_i} g_i(z)} (z - z_i) = r_i.$$

By the residue theorem we have

$$\operatorname{Re}\left(\int_{|z|=1} \frac{f'}{f}\, dz\right) = \operatorname{Re}\left(2\pi\sqrt{-1} \sum_{i=1}^{n} r_i\right) = 0.$$

Problem 9. Let f be a holomorphic function on $\{z \in \mathbb{C},\ 0 < |z| < 1\}$. Suppose there are two constants M and A such that

$$|f(z)| \leq M + A|z|^{-3/4} \quad \text{for } 0 < |z| < 1.$$

Show that f has a removable singularity at $z = 0$.

Solution to Problem 9. Let $g(z) = zf(z)$. Then

$$|g(z)| \leq M|z| + A|z|^{1/4}$$

and g is a bounded holomorphic function on the unit disc. Hence $g(z)$ has a removable singularity at $z = 0$. Since $g(0) = 0$, we can write in a neighborhood of 0 that $g(z) = z \cdot g_1(z)$ for some holomorphic function $g_1(z)$. Hence $f(z) = g_1(z)$ and f has a removable singularity at 0.