ANALYSIS QUALIFYING EXAM FOR FALL 2007

Problem 0.1. Evaluate
$$\lim_{n\to\infty}\int_0^\infty x^{-2/3}e^{-nx^2}\,dx.$$

Problem 0.2. Let μ be a finite Borel measure on $[0,1]$. Prove that there exists $t\in[0,1]$ such that
$$\int_{[0,1]}\frac{1}{\sqrt{|t-x|}}\,d\mu(x)<\infty.$$

Problem 0.3. For $n\in\mathbb{N}$ let $f_n\colon[0,1]\to[0,\infty)$ be a measurable function. Show that there are $\alpha_n>0$ and a set $E\subset[0,1]$ with measure zero such that $\sum_{n=1}^\infty \alpha_n f_n(x)$ converges for all $x\in[0,1]\setminus E$.

Problem 0.4. Let E be a Banach space. Let $L(E)$ be the vector space of all continuous linear maps from E to E, with the usual operator norm. Prove that $L(E)$ is complete.

Problem 0.5. Let $f\colon(0,\infty)\to\mathbb{C}$ be continuous and have compact support. Define $g\colon\mathbb{R}\to\mathbb{C}$ by
$$g(t)=\int_0^\infty f(x)e^{itx^7}\,dx.$$
Prove that $g\in L^2(\mathbb{R})$.

Problem 0.6. Set $\Omega=\{z\in\mathbb{C}\colon |z|<3\}$. Let A be the vector space of all $f\in C(\overline{\Omega})$ such that $f|_\Omega$ is holomorphic, with the norm from $C(\overline{\Omega})$. Define a linear functional $\omega\colon A\to\mathbb{C}$ by
$$\omega(f)=\int_0^{4\pi}\frac{f(2e^{it})ie^{it}}{e^{it}-1}\,dt.$$
Prove that ω is bounded and find $\|\omega\|$.

Problem 0.7. Let $\Omega=\{z\in\mathbb{C}\colon |z|<2\}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on Ω such that $\sup_{n\in\mathbb{N}}\sup_{z\in\Omega}|f_n(z)|<\infty$ and $\lim_{n\to\infty}f_n(1/k)=0$ for all $k\in\mathbb{N}$. Prove that $\lim_{n\to\infty}f_n(0)=0$.

Problem 0.8. Let f be the meromorphic function on $\mathbb{C}$ given by
$$f(z)=\frac{e^z}{z(z-2)}+\frac{1}{(z-4)^2}.$$
Define $\gamma\colon[0,2\pi]\to\mathbb{C}$ by $\gamma(t)=4-3e^{it}$. Evaluate $\int_\gamma f(z)\,dz$.

Problem 0.9. Define $f\colon\mathbb{C}\to\mathbb{C}$ by $f(z)=z^6+2007z^3+z-2$. How many zeros (counting multiplicity) does f have in the open unit disk?

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ANALYSIS QUALIFYING EXAM SOLUTIONS FOR FALL 2007

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Problem 0.1. Evaluate

$$\lim_{n\to\infty} \int_0^\infty x^{-2/3} e^{-nx^2}\, dx.$$

Solution. Set $f_n(x) = x^{-2/3}e^{-nx^2}$ for $n \in \mathbb{N}$ and $x \in (0, \infty)$. Clearly $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0, \infty)$. We show that the integrals converge by using the Dominated Convergence Theorem.

Define $g\colon (0, \infty) \to [0, \infty)$ by $g(x) = x^{-2/3}e^{-x^2}$. We claim that g is integrable. We calculate:

$$\begin{aligned} \int_0^\infty g\, dm &= \int_0^1 x^{-2/3}e^{-x^2}\, dx + \int_1^\infty x^{-2/3}e^{-x^2}\, dx \\ &\leq \int_0^1 x^{-2/3}\, dx + \int_0^\infty e^{-x^2}\, dx = 3 + \tfrac{1}{2}\sqrt{\pi}, \end{aligned}$$

proving the claim.

Next, observe that $-nx^2 \leq -x^2$ for all $x \in \mathbb{R}$ and $n \geq 1$, from which it follows that $f_n(x) \leq g(x)$.

The Dominated Convergence Theorem now implies that

$$\lim_{n\to\infty} \int_0^\infty x^{-2/3}e^{-nx^2}\, dx = \int_0^\infty \left(\lim_{n\to\infty} x^{-2/3}e^{-nx^2} \right) dx = \int_0^\infty 0\, dx = 0.$$

This completes the solution. □

Problem 0.2. Let μ be a finite Borel measure on $[0, 1]$. Prove that there exists $t \in [0, 1]$ such that

$$\int_{[0,1]} \frac{1}{\sqrt{|t - x|}}\, d\mu(x) < \infty.$$

Solution. Define $f\colon [0, 1] \times [0, 1] \to [0, \infty)$ by

$$f(x, y) = \begin{cases} \frac{1}{\sqrt{|y-x|}} & y \neq x \\ 0 & y = x. \end{cases}$$

Then f is Borel on $\{(x, y) \in [0, 1]^2 \colon x \neq y\}$ because f is continuous there, and it follows easily that f is Borel on $[0, 1] \times [0, 1]$. Therefore we may apply Fubini's Theorem for nonnegative functions. With m being Lebesgue measure, the first step being Fubini's Theorem, and the second step following from $[0, 1] \subset [x - 1,\, x + 1]$

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when $x \in [0,1]$, we get

$$\begin{aligned}\int_0^1 \left(\int_0^1 f(x,t)\, d\mu(x) \right) dm(t) &= \int_0^1 \left(\int_0^1 f(x,t)\, dm(t) \right) d\mu(x) \\ &\leq \int_0^1 \left(\int_{x-1}^{x+1} \frac{1}{\sqrt{|t-x|}}\, dm(t) \right) d\mu(x) \\ &= \int_0^1 4\, d\mu(x) = 4\mu([0,1]) < \infty.\end{aligned}$$

Therefore there exists $t \in [0,1]$ such that the integrand in the first expression is finite. (In fact, this is true for almost every $t \in [0,1]$ with respect to Lebesgue measure.) □

Problem 0.3. For $n \in \mathbb{N}$ let $f_n \colon [0,1] \to [0,\infty)$ be a measurable function. Show that there are $\alpha_n > 0$ and a set $E \subset [0,1]$ with measure zero such that $\sum_{n=1}^{\infty} \alpha_n f_n(x)$ converges for all $x \in [0,1] \setminus E$.

This problem is from the Purdue graduate exam in measure theory from January 1997.

Solution. For each n, the sets

$$F_{k,n} = \{x \in [0,1] \colon f_n(x) > k\}$$

are measurable sets which decrease with k and satisfy $\bigcap_{k=1}^{\infty} F_{k,n} = \varnothing$. Since $[0,1]$ has finite measure, there is $t(n)$ such that $m(F_{t(n),n}) < 2^{-n}$. Set

$$\alpha_n = \frac{1}{n^2 t(n)}$$

for $n \in \mathbb{N}$, and set

$$E = \bigcap_{l=1}^{\infty} \bigcup_{n=l}^{\infty} F_{t(n),n}.$$

Suppose $x \notin E$. Then there is l such that $x \notin \bigcup_{n=l}^{\infty} F_{t(n),n}$. So $f_n(x) \leq t(n)$ for all $n \geq l$, whence $\alpha_n f_n(x) \leq \frac{1}{n^2}$ for all $n \geq l$. It follows that $\sum_{n=1}^{\infty} \alpha_n f_n(x)$ converges.

We finish by showing that $m(E) = 0$. The sets $E_l = \bigcup_{n=l}^{\infty} F_{t(n),n}$ decrease with l and satisfy

$$m(E_l) \leq \sum_{n=l}^{\infty} m(F_{t(n),n}) < \sum_{n=l}^{\infty} \frac{1}{2^n} = \frac{1}{2^{l-1}}.$$

So $\lim_{l \to \infty} m(E_l) = 0$, whence $m(E) = 0$. □

Problem 0.4. Let E be a Banach space. Let $L(E)$ be the vector space of all continuous linear maps from E to E, with the usual operator norm. Prove that $L(E)$ is complete.

At least half the credit will be lost if the last step (proving that $\|a_n - a\|$ actually converges to zero) is omitted.

Solution. Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $L(E)$. For every $\xi \in E$, we have

$$\|a_m \xi - a_n \xi\| = \|(a_m - a_n)\xi\| \leq \|a_m - a_n\| \cdot \|\xi\|.$$

Therefore the sequence $(a_n\xi)_{n\in\mathbb{N}}$ is Cauchy in E. So it has a limit $a(\xi) \in E$, depending on ξ.

We claim that $a\colon E \to E$ is linear. Let $\xi, \eta \in E$ and $\alpha, \beta \in \mathbb{C}$. Then

$$a(\alpha\xi + \beta\mu) = \lim_{n\to\infty} a_n(\alpha\xi + \beta\mu) = \lim_{n\to\infty} (\alpha a_n\xi + \beta a_n\mu) = \alpha a\xi + \beta a\mu.$$

This proves the claim.

We claim that $a\colon E \to E$ is bounded. Since $|\|a_m\| - \|a_m\|| \le \|a_m - a_n\|$, the sequence $(\|a_n\|)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Therefore this sequence is bounded. Set $M = \sup_{n\in\mathbb{N}} \|a_n\| < \infty$. Then for all $\xi \in E$, we have

$$\|a\xi\| = \lim_{n\to\infty} \|a_n\xi\| \le \limsup_{n\to\infty} \|a_n\| \cdot \|\xi\| \le M\|\xi\|.$$

This proves the claim. We now know that $a \in L(E)$.

Finally, we prove that $\lim_{n\to\infty} \|a_n - a\| = 0$. Let $\varepsilon > 0$, and choose N such that $m, n \ge N$ imply $\|a_m - a_n\| < \frac{1}{2}\varepsilon$. For $\xi \in E$ and $n \ge N$, we have

$$\|(a_n - a)\xi\| = \|a_n\xi - a\xi\| = \lim_{m\to\infty} \|a_n\xi - a_m\xi\| \le \limsup_{m\to\infty} \|a_m - a_n\| \cdot \|\xi\| \le \tfrac{1}{2}\varepsilon\|\xi\|.$$

Therefore $n \ge N$ implies $\|a_n - a\| \le \frac{1}{2}\varepsilon < \varepsilon$. This proves the claim. Thus, $\lim_{n\to\infty} a_n = a$, and we have finished the proof that $L(E)$ is complete. □

Problem 0.5. Let $f\colon (0,\infty) \to \mathbb{C}$ be continuous and have compact support. Define $g\colon \mathbb{R} \to \mathbb{C}$ by

$$g(t) = \int_0^\infty f(x)e^{itx^7}\,dx.$$

Prove that $g \in L^2(\mathbb{R})$.

The factor $\sqrt{2\pi}$ in the equation $g(t) = \sqrt{2\pi}\,\widehat{h}(t)$ in the solution is from Rudin's choice of the normalization of the Fourier transform.

Solution. Using the change of variables $y = x^7$, giving $dx = \frac{1}{7}y^{-6/7}\,dy$, we get

$$g(t) = \int_0^\infty f(x)e^{itx^7}\,dx = \int_0^\infty f\big(y^{1/7}\big)e^{ity}\tfrac{1}{7}y^{-6/7}\,dy.$$

(There is no issue at 0 because the integrand vanishes on a neighborhood of 0.) Thus, $g(t) = \sqrt{2\pi}\,\widehat{h}(t)$ with

$$h(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{7}y^{-6/7}f\big(y^{1/7}\big) & y > 0. \end{cases}$$

We have $h \in L^2(\mathbb{R})$ because h vanishes off $[a,b]$ for some a and b with $0 < a < b < \infty$. So $\widehat{h} \in L^2(\mathbb{R})$. □

Problem 0.6. Set $\Omega = \{z \in \mathbb{C}\colon |z| < 3\}$. Let A be the vector space of all $f \in C\big(\overline{\Omega}\big)$ such that $f|_\Omega$ is holomorphic, with the norm from $C\big(\overline{\Omega}\big)$. Define a linear functional $\omega\colon A \to \mathbb{C}$ by

$$\omega(f) = \int_0^{4\pi} \frac{f\big(2e^{it}\big)ie^{it}}{e^{it} - 1}\,dt.$$

Prove that ω is bounded and find $\|\omega\|$.

Solution. Define $\gamma\colon [0, 4\pi] \to \mathbb{C}$ by $\gamma(t) = 2e^{it}$. Then for $f \in A$, we have

$$|\omega(f)| = \left| \int_\gamma \frac{f(z)}{z-1}\, dz \right| = \left| 2\pi i \operatorname{Ind}_\gamma(1) f(1) \right| = 2\pi \cdot 2 \cdot |f(1)| \leq 4\pi \|f\|_\infty.$$

So $\|\omega\| \leq 4\pi$.

If we take $f(z) = 1$ for all $z \in \overline{\Omega}$, then $\|f\|_\infty = 1$ and $\omega(f) = 4\pi i f(1) = 4\pi i$. This shows $\|\omega\| \geq 4\pi$. So $\|\omega\| = 4\pi$. □

Problem 0.7. Let $\Omega = \{z \in \mathbb{C}\colon |z| < 2\}$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of holomorphic functions on Ω such that $\sup_{n \in \mathbb{N}} \sup_{z \in \Omega} |f_n(z)| < \infty$ and $\lim_{n \to \infty} f_n(1/k) = 0$ for all $k \in \mathbb{N}$. Prove that $\lim_{n \to \infty} f_n(0) = 0$.

Solution. Set $M = \sup_{n \in \mathbb{N}} \sup_{z \in \Omega} |f_n(z)|$. We claim that $|f_n'(z)| \leq M$ for $n \in \mathbb{N}$ and $|z| \leq 1$. Indeed, for each such n and z, the disk $B = \{w \in \mathbb{C}\colon |w - z| < 1\}$ is contained in Ω, and $|f_n| \leq M$ on B, so the claim follows from Cauchy's Estimates.

The Mean Value Theorem for vector valued functions on an interval in $\mathbb{R}$ now implies that $|f_n(b) - f_n(a)| \leq M|b - a|$ for $a, b \in [0, 1]$.

Now let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $M/k < \frac{1}{2}\varepsilon$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(1/k)| < \frac{1}{2}\varepsilon$. For $n \geq N$, we then have

$$|f_n(0)| \leq |f_n(0) - f_n(1/k)| + |f_n(1/k)| \leq M(1/k) + |f_n(1/k)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof. □

(One can also get an estimate similar to the one in the first paragraph by estimating Cauchy's formula for $f_n'(z)$. Unfortunately, this theorem is not in Rudin's book.)

Problem 0.8. Let f be the meromorphic function on $\mathbb{C}$ given by

$$f(z) = \frac{e^z}{z(z-2)} + \frac{1}{(z-4)^2}.$$

Define $\gamma\colon [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = 4 - 3e^{it}$. Evaluate $\int_\gamma f(z)\, dz$.

Solution. Clearly γ is a closed curve. The Residue Theorem tells us that

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

is the sum of the residues of f at its poles, each multiplied by the winding number of γ about the corresponding pole. The poles of f are at 0, 2, and 4. A direct computation shows that the winding number of γ about 4 is -1. (This is geometrically obvious, and I don't expect the calculation to be carried out. However, something must be said.) The point 2 is in the same connected component of $\mathbb{C} \setminus \gamma([0, 2\pi])$, so the winding number of γ about 1 is also -1. Also 0 is in the unbounded component of $\mathbb{C} \setminus \gamma([0, 2\pi])$, so the winding number of γ about 0 is 0. (Again, something must be said, but not much.)

We need only find the residues at 2 and 4. The residue at 2 is

$$\lim_{z \to 2} (z - 2) f(z) = \lim_{z \to 2} \left(\frac{e^z}{z} + \frac{z-2}{(z-4)^2} \right) = \tfrac{1}{2} e^2.$$

(This computation gives the residue because the limit exists. There are also other ways to get it.) The residue at 4 is the same as the residue of $(z-4)^{-2}$ at 4, because

$f(z) - (z-4)^{-2}$ is holomorphic on a neighborhood of 4. The residue of $(z-4)^{-2}$ at 4 is clearly zero, because the Laurent series about 4 has no term with $(z-4)^{-1}$.

It follows that

$$\int_\gamma f(z)\,dz = 2\pi i\left(\tfrac{1}{2}e^2\right) = \pi i e^2.$$

This completes the solution. □

Problem 0.9. Define $f\colon \mathbb{C} \to \mathbb{C}$ by $f(z) = z^6 + 2007z^3 + z - 2$. How many zeros (counting multiplicity) does f have in the open unit disk?

Rouché's Theorem was done in the course, but not heavily emphasized. However, there has been at least one recent problem using Rouché's Theorem.

Solution. Set $g(z) = 2007z^3$. For $|z| = 1$, we have

$$|f(z) - g(z)| = |z^6 + z - 2| \leq 4 < 2007 = |g(z)|.$$

Therefore Rouché's Theorem implies that f and g have the same number of zeros (counting multiplicity) in the open unit disk. Obviously g has 3 zeros (counting multiplicity) in the open unit disk. So the same is true of f. □