Analysis Qualifying Exam, Fall 2006

1. Suppose that μ is a positive Borel measure on $\mathbb{R}^2$ such that

$$\mu(L) > 0 \qquad \text{for any straight line } L \subset \mathbb{R}^2.$$

Prove that $\mu(\mathbb{R}^2) = \infty$.

2. Give an example of a nowhere dense set $E \subset [0,1]$ such that $|E| > 0$.

3. Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^n$. Prove that there exists a Borel function g on $\mathbb{R}^n$ such that $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.

4. Let $f \in L^1(\mathbb{R})$ be compactly supported. Prove that

$$g(z) = \int_{-\infty}^{\infty} f(x) e^{xz} dx$$

is an entire function. Conclude that $\hat{f}(\xi)$ has at most countably many zeros.

5. Let $\mathcal{H}$ be a Hilbert space. We say that a sequence $\{x_n\} \subset \mathcal{H}$ converges weakly to x if

$$\langle x_n, y \rangle \to \langle x, y \rangle \qquad \text{for all } y \in \mathcal{H}.$$

Suppose that $\{T_n\}$ is a sequence of bounded linear operators on $\mathcal{H}$ such that that for all $x \in \mathcal{H}$, $\{T_n x\}$ converges weakly to some $y = Tx$. Prove that T is a bounded linear operator on $\mathcal{H}$.

6. Suppose that Λ is a bounded linear functional on $C_0(\mathbb{R})$. Prove that the limit

$$\lim_{n\to\infty} \Lambda(f_n), \qquad f_n(x) = e^{-n|x|}$$

exists. Describe when it is non-zero.

7. Let $f = \chi_{[0,1]}$. Prove that its maximal function

$$Mf(x) = \sup_{I \text{ interval s.t. } x \in I} \frac{1}{|I|} \int_I |f|$$

is not in $L^1(\mathbb{R})$.

8. Suppose that f is holomorphic on $\mathbb{C} \setminus \{0\}$ and there exists $N \in \mathbb{N}$ such that

$$|f(z)| \leq C|z|^{-N} \qquad \text{for all } z \neq 0.$$

Prove that f is a rational functional with a pole at 0 of order at most N.

9. Let $C = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that f is a holomorphic function on $\mathbb{C} \setminus C$ and continuous on $\mathbb{C}$. Prove that f is entire.