Analysis Qualifying Examination

Fall 2005

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let E_n be a sequence of measurable sets. Suppose that $\lim_{n\to\infty} \mu(E_n) = 0$. Show that there exists a subsequence $\{n_k\}$ such that

$$\mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_{n_k})) = 0.$$

2. Let $\{f_n\}$ be a sequence of bounded functions in $L^1([0,\pi])$ such that $\lim_{n\to\infty} \|f_n\|_\infty = 0$. Find

$$\lim_{n\to\infty} \int_0^\pi n f_n(t) e^{-n \sin t} dt.$$

(Show your work!)

3. Let f be a real measurable function on a compact metric space X and μ be a finite positive Borel measure on X. Suppose that

$$\int_O f d\mu = 0$$

for all open subset $O \subset X$. Show that $f(x) = 0$ a.e. μ.

4. Let f_n be a sequence of functions on $[0,1]$. Suppose that each f_n has bounded variation on $[0,1]$. Suppose that f_n converges uniformly to a function f on $[0,1]$. Does f have bounded variation? Why?

5. Let $\{x_n\}$ be a complex sequence. Suppose that for any $\{y_n\} \in l^1$, $\{x_n y_n\} \in l^1$. Show that $\{x_n\}$ is a bounded sequence.

6. Define an operator T on $C([0,1])$ by $T(f)(x) = \int_0^x f(t)dt$. Show that T maps $C([0,1])$ into $C([0,1])$. Find $\lim_{n\to\infty} \|T^n\|$.

7. Let H be the set of functions which are holomorphic on an open set which contains the closed unit disk. View H as a subspace of $C(\overline{D})$. Show that H is not dense in $C(\overline{D})$, where D is the unit disk.

8. Let f be an entire function with finitely many zeros. Suppose that $f(1/z)$ has a pole of order at least two. Show that the sum of the residues of $\frac{1}{f(z)}$ at all zeros of f must be zero.

9. Suppose that f is holomorphic on $\Omega = \{z \in \mathbf{C} : |z| > 1/2\}$. Suppose that $\lim_{z\to\infty} |f(z)/z| = 0$. Show that f is bounded on $A = \{z \in \mathbf{C} : |z| > 1\}$.