ANALYSIS QUALIFYING EXAM FALL 2004

1. Let (X, μ) be a measure space, let $1 \leq p_1 < p < p_2 < \infty$, and let $f \colon X \to \mathbf{C}$ be a measurable function such that $\|f\|_{p_1} < \infty$ and $\|f\|_{p_2} < \infty$. Prove that $\|f\|_p < \infty$.

2. Determine, with justification, the limit

$$\lim_{n\to\infty} \int_0^n \frac{\cos(x/n)}{\sqrt{x+\cos(x/n)}}\, dx.$$

3. Let X be a compact metric space, and let ω be a positive linear functional on $C(X)$. Let $f \in C(X)$, and let $(f_n)_{n\in\mathbf{N}}$ be a bounded sequence in $C(X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Prove that $\lim_{n\to\infty} \omega(f_n) = \omega(f)$.

4. Let H be a Hilbert space. Let $(\xi_n)_{n\in\mathbf{N}}$ be a sequence in H, and let $\xi \in H$. Suppose that $\lim_{n\to\infty}\langle \xi_n, \eta\rangle = \langle \xi, \eta\rangle$ for all $\eta \in H$, and that $\lim_{n\to\infty} \|\xi_n\| = \|\xi\|$. Prove that $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

5. Let E and F be Banach spaces. Let V be the vector space of all pairs (ξ, η) with $\xi \in E_1$ and $\eta \in E_2$. For $(\xi, \eta) \in E$, define $\|(\xi, \eta)\| = \|\xi\| + \|\eta\|$. Prove that $\|\cdot\|$ is a norm on E, and that E is complete.

6. Let E be a Banach space, and let $F \colon \mathbf{C} \to E$ be a continuous function. Suppose that for every $\omega \in E^*$, the function $z \mapsto \omega(F(z))$ is holomorphic. Suppose that for every $\varepsilon > 0$ there is a compact set $K \subset \mathbf{C}$ such that $z \notin K$ implies $\|F(z)\| < \varepsilon$. Prove that $F = 0$.

7. Let f be a holomorphic function on $\Omega = \{z \in \mathbf{C} \colon |z| < 1\}$. Suppose that $f(z)$ is purely imaginary for $z \in \Omega$ real. Prove that $f(\overline{z}) = -\overline{f(z)}$ for all $z \in \Omega$.

8. Let a, b, $c \in \mathbf{C}$ be constants. Let f be the meromorphic function on $\mathbf{C}$ given by

$$f(z) = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{z-4}.$$

Evaluate $\int_\gamma f(z)\, dz$, where $\gamma \colon [0, 2\pi] \to \mathbf{C}$ is given by $\gamma(t) = 2e^{it}$.

9. Let $\Omega \subset C$ be open, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$. Suppose that there are a neighborhood U of a and constants M and c such that $|f(z)| \leq M + c|z-a|^{-1/2}$ for $z \in U \setminus \{a\}$. Prove that f has a removable singularity at a.

SOLUTIONS TO ANALYSIS QUALIFYING EXAM FALL 2004

Difficulty ratings are guesses, and are on a scale of 1 (easy) to 3 (hard).

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1. Let (X, μ) be a measure space, let $1 \leq p_1 < p < p_2 < \infty$, and let $f \colon X \to \mathbf{C}$ be a measurable function such that $\|f\|_{p_1} < \infty$ and $\|f\|_{p_2} < \infty$. Prove that $\|f\|_p < \infty$.

Comment: Difficulty rating: 1.

This problem is from the Purdue graduate exam in measure theory from August 1994. It is an excellent short example of partitioning a set and using different estimates on the parts.

Solution: Set $E = \{x \in X \colon |f(x)| > 1\}$. For $x \in E$ we have $|f(x)|^p \leq |f(x)|^{p_1}$ and for $x \in X \setminus E$ we have $|f(x)|^p \leq |f(x)|^{p_2}$. Therefore

$$\int_X |f|^p \, d\mu = \int_E |f|^p \, d\mu + \int_{X \setminus E} |f|^p \, d\mu \leq \int_E |f|^{p_1} \, d\mu + \int_{X \setminus E} |f|^{p_2} \, d\mu$$
$$\leq \int_X |f|^{p_1} \, d\mu + \int_X |f|^{p_2} \, d\mu = \|f\|_{p_1}^{p_1} + \|f\|_{p_2}^{p_2} < \infty.$$

Thus $\|f\|_p < \infty$. ∎

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2. Determine, with justification, the limit

$$\lim_{n \to \infty} \int_0^n \frac{\cos(x/n)}{\sqrt{x + \cos(x/n)}} \, dx.$$

Comment: Difficulty rating: 1.5.

Solution: Set

$$f_n(x) = \frac{\chi_{[0,n]}(x) \cos(x/n)}{\sqrt{x + \cos(x/n)}}.$$

Then $\lim_{n \to \infty} f_n(x) = (x+1)^{-1/2}$ for all $x \in [0, \infty)$. We apply Fatou's Lemma:

$$\liminf_{n \to \infty} \int_0^\infty f_n \geq \int_0^\infty \lim_{n \to \infty} f_n = \int_0^\infty \frac{1}{\sqrt{x+1}} \, dx = \infty.$$

Therefore $\lim_{n \to \infty} \int_0^\infty f_n$ exists and is equal to ∞. ∎

Alternate solution: Set

$$f_n(x) = \frac{\chi_{[0,n]}(x) \cos(x/n)}{\sqrt{x + \cos(x/n)}}.$$

For $x \in [0, n]$ we have

$$1 \geq \cos(x/n) \geq \cos(1) > 0.$$

Therefore

$$f_n(x) \geq \frac{\chi_{[0,n]}(x)\cos(1)}{\sqrt{x+1}}$$

for all n and x. Now

$$\int_0^\infty \frac{\chi_{[0,n]}(x)\cos(1)}{\sqrt{x+1}}\,dx = \int_0^n \frac{\cos(1)}{\sqrt{x+1}}\,dx = 2\cos(1)\left(\sqrt{n+1}-1\right).$$

So

$$\int_0^\infty f_n \geq 2\cos(1)\left(\sqrt{n+1}-1\right).$$

Since

$$\lim_{n\to\infty} 2\cos(1)\left(\sqrt{n+1}-1\right) = \infty,$$

it follows that $\lim_{n\to\infty}\int_0^\infty f_n$ exists and is equal to ∞. ∎

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3. Let X be a compact metric space, and let ω be a positive linear functional on $C(X)$. Let $f \in C(X)$, and let $(f_n)_{n\in\mathbf{N}}$ be a bounded sequence in $C(X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Prove that $\lim_{n\to\infty}\omega(f_n) = \omega(f)$.

Comment: Difficulty rating: 1.

Solution: According to the Riesz Representation Theorem, there is a (positive) Borel measure μ on X such that $\omega(g) = \int_X g\,d\mu$ for all $g \in C(X)$. The measure μ is finite because $\mu(X) = \omega(1) < \infty$. Set $M = \sup_{n\in\mathbf{N}}\|f_n\|$. Then the constant function M is integrable, and $|f_n| \leq M$ for all n, so the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty}\omega(f_n) = \lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu = \omega(f),$$

as desired. ∎

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4. Let H be a Hilbert space. Let $(\xi_n)_{n\in\mathbf{N}}$ be a sequence in H, and let $\xi \in H$. Suppose that $\lim_{n\to\infty}\langle\xi_n,\eta\rangle = \langle\xi,\eta\rangle$ for all $\eta \in H$, and that $\lim_{n\to\infty}\|\xi_n\| = \|\xi\|$. Prove that $\lim_{n\to\infty}\|\xi_n - \xi\| = 0$.

Comment: Difficulty rating: 1.

Solution: We have

$$\begin{aligned}
\lim_{n\to\infty}\|\xi_n-\xi\|^2 &= \lim_{n\to\infty}\langle\xi_n-\xi,\,\xi_n-\xi\rangle \\
&= \lim_{n\to\infty}\langle\xi_n,\xi_n\rangle - \lim_{n\to\infty}\langle\xi_n,\xi\rangle - \lim_{n\to\infty}\overline{\langle\xi_n,\xi\rangle} + \lim_{n\to\infty}\langle\xi,\xi\rangle \\
&= \lim_{n\to\infty}\|\xi_n\|^2 - \lim_{n\to\infty}\langle\xi_n,\xi\rangle - \overline{\lim_{n\to\infty}\langle\xi_n,\xi\rangle} + \|\xi\|^2 \\
&= \|\xi_n\|^2 - \langle\xi,\xi\rangle - \overline{\langle\xi,\xi\rangle} + \|\xi\|^2 \\
&= 0,
\end{aligned}$$

as desired. ∎

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5. Let E and F be Banach spaces. Let V be the vector space of all pairs (ξ, η) with $\xi \in E_1$ and $\eta \in E_2$. For $(\xi, \eta) \in E$, define $\|(\xi, \eta)\| = \|\xi\| + \|\eta\|$. Prove that $\|\cdot\|$ is a norm on E, and that E is complete.

Comment: Difficulty rating: 1.

Solution: The verifications that $\|(\xi, \eta)\| \geq 0$, that $\|\lambda(\xi, \eta)\| = |\lambda| \cdot \|(\xi, \eta)\|$, and that $\|(\xi_1, \eta_1) + (\xi_2, \eta_2)\| \leq \|(\xi_1, \eta_1)\| + \|(\xi_2, \eta_2)\|$, are completely routine, and are omitted from the solution.

Now suppose $\|(\xi, \eta)\| = 0$. Then $\|\xi\| + \|\eta\| = 0$. Because $\|\xi\| \geq 0$ and $\|\eta\| \geq 0$, we get $\|\xi\| = \|\eta\| = 0$. So $\xi = 0$ and $\eta = 0$, whence $(\xi, \eta) = 0$.

It remains to prove completeness. Let $((\xi_n, \eta_n))_{n \in \mathbf{N}}$ be a Cauchy sequence in V. Since $\|(\xi, \eta)\| \geq \max(\|\xi\|, \|\eta\|)$ for any $\xi \in E$ and $\eta \in F$, the sequences $(\xi_n)_{n \in \mathbf{N}}$ and $(\eta_n)_{n \in \mathbf{N}}$ are Cauchy in E and F respectively. Therefore they have limits $\xi \in E$ and $\eta \in F$.

We prove that $\lim_{n \to \infty} \|(\xi, \eta) - (\xi_n, \eta_n)\| = 0$. Since

$$\lim_{n \to \infty} \|\xi - \xi_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\eta - \eta_n\| = 0,$$

it follows that

$$\lim_{n \to \infty} (\|\xi - \xi_n\| + \|\eta - \eta_n\|) = 0,$$

as desired. ∎

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6. Let E be a Banach space, and let $F \colon \mathbf{C} \to E$ be a continuous function. Suppose that for every $\omega \in E^*$, the function $z \mapsto \omega(F(z))$ is holomorphic. Suppose that for every $\varepsilon > 0$ there is a compact set $K \subset \mathbf{C}$ such that $z \notin K$ implies $\|F(z)\| < \varepsilon$. Prove that $F = 0$.

Comment: Difficulty rating: 1.
Could also use the condition $\lim_{|z| \to \infty} \omega(F(z)) = 0$ for all $\omega \in B^*$.

Solution: For $\omega \in B^*$ define $F_\omega(z) = \omega(F(z))$. Then F_ω is an entire function which vanishes at infinity, so Liouville's Theorem implies that F_ω is constant, and therefore zero. Thus, $\omega(F(z)) = 0$ for every $\omega \in E^*$ and every $z \in \mathbf{C}$. The Hahn-Banach Theorem now implies $F(z) = 0$ for every $z \in \mathbf{C}$. ∎

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7. Let f be a holomorphic function on $\Omega = \{z \in \mathbf{C} \colon |z| < 1\}$. Suppose that $f(z)$ is purely imaginary for $z \in \Omega$ real. Prove that $f(\overline{z}) = -\overline{f(z)}$ for all $z \in \Omega$.

Comment: Difficulty rating: 2.
Problems like this sometimes occur in homework, but not this year.

Solution: Define $g \colon \Omega \to \mathbf{C}$ by $g(z) = \overline{f(\overline{z})}$. We claim that g is holomorphic, with derivative $g'(z) = \overline{f'(\overline{z})}$. So let $z \in \Omega$, let $\varepsilon > 0$, and choose $\delta > 0$ such that if

$0 < |h| < \delta$ then

$$\left| \frac{f(\overline{z} + h) - f(\overline{z})}{h} - f'(\overline{z}) \right| < \varepsilon.$$

Then $0 < |h| < \delta$ implies

$$\left| \frac{g(z + h) - g(z)}{h} - \overline{f'(\overline{z})} \right| = \left| \frac{\overline{f(\overline{z + h})} - \overline{f(\overline{z})}}{h} - \overline{f'(\overline{z})} \right|$$

$$= \left| \overline{\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}} - f'(\overline{z})} \right| < \varepsilon,$$

because $0 < |\overline{h}| < \delta$. This proves the claim.

We have $g(z) = -f(z)$ for all $z \in \Omega \cap \mathbf{R}$. Since this set has a cluster point in Ω, and since Ω is connected, it follows that $g = -f$, which implies the result. ∎

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8. Let a, b, $c \in \mathbf{C}$ be constants. Let f be the meromorphic function on $\mathbf{C}$ given by

$$f(z) = \frac{a}{z} + \frac{b}{z - 1} + \frac{c}{z - 4}.$$

Evaluate $\int_\gamma f(z)\, dz$, where $\gamma\colon [0, 2\pi] \to \mathbf{C}$ is given by $\gamma(t) = 2e^{it}$.

Comment: Difficulty rating: 1.

Solution: Clearly γ is a closed curve. The Residue Theorem tells us that

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

is the sum of the residues of f at its poles, each multiplied by the winding number of γ about the corresponding pole. The poles of f are only at 0, 1, and 4. (Some of these might not be poles, since some of the constants a, b, and c might be zero.) A direct computation shows that the winding number of γ about 0 is 1. (This is geometrically obvious, and I don't expect the calculation to be carried out. However, something must be said.) The point 1 is in the same connected component of $\mathbf{C} \setminus \gamma([0, 2\pi])$, so the winding number of γ about 1 is also 1. Also 4 is in the unbounded component of $\mathbf{C} \setminus \gamma([0, 2\pi])$, so the winding number of γ about 4 is 0.

We need only find the residues at 0 and 1. These are evidently a and b respectively. (For example, the residue at 0 is the coefficient of z^{-1} in the Laurent series of f about 0. The terms $\frac{b}{z-1}$ and $\frac{c}{z-4}$ are holomorphic in a neighborhood of 0, so contribute no negative powers of z. Thus the coefficient of z^{-1} is a.)

It follows that

$$\int_\gamma f(z)\, dz = 2\pi i(a + b).$$

∎

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9. Let $\Omega \subset C$ be open, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$. Suppose that there are a neighborhood U of a and constants M and c such that

$|f(z)| \leq M + c|z-a|^{-1/2}$ for $z \in U \setminus \{a\}$. Prove that f has a removable singularity at a.

Comment: Difficulty rating: 1.

Solution: Set $g_0(z) = (z-a)f(z)$. Then g_0 is holomorphic function on $\Omega \setminus \{a\}$ and $\lim_{z \to a} g_0(z) = 0$, so g has a removable singularity at a. Thus there is a holomorphic function g on Ω such that $g|_{\Omega \setminus \{a\}} = g_0$. By continuity we have $g(a) = 0$. Since g has a zero of order at least one at a, there is a holomorphic function h on Ω such that $h(z) = g(z)/(z-a) = f(z)$ for all $z \in \Omega \setminus \{a\}$. The existence of h shows that f has a removable singularity at a. ∎

Alternate solution: Set $g(z) = (z-a)^2 f(z)$ for $z \in \Omega \setminus \{a\}$, and set $g(a) = 0$. Then $g'(z)$ clearly exists for every $z \in \Omega \setminus \{a\}$.

We claim that $g'(a) = 0$. Indeed, $z \in \Omega \setminus \{a\}$ we have

$$\begin{aligned} 0 \leq \left| \frac{g(z) - g(a)}{z-a} \right| &= \left| \frac{(z-a)^2 f(z)}{z-a} \right| \\ &\leq \frac{|z-a|^2 (M + c|z-a|^{-1/2})}{|z-a|} = M|z-a| + c|z-a|^{1/2}. \end{aligned}$$

Therefore

$$g'(a) = \lim_{z \to a} \frac{g(z) - g(a)}{z-a} = 0,$$

proving the claim.

It follows that g is holomorphic on Ω. Since $g(a) = g'(a) = 0$, the function g has a zero of order at least 2 at a. So there is a holomorphic function h on Ω such that $h(z) = g(z)/(z-a)^2 = f(z)$ for all $z \in \Omega \setminus \{a\}$. The existence of h shows that f has a removable singularity at a. ∎