Analysis Qualifying Examination

Fall 2003

Instruction: Partial credit will be given when appropriate. The decision on this examination will place emphasis not only on the total point score, but also on whether the answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot provided.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

1. Let f be an entire fucntion such that the infinity is a pole of order k. Show that f is a polynomial.

2. Let U be the open unit disk, let $f \in H(U)$ and $f \in C(\overline{U})$. Prove that

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz,$$

where $|a| < 1$ and γ is the unit circle.

3. Compute

$$\int_0^\infty \frac{dx}{1+x^{10}}.$$

4. Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{1-\cos(x/n)}{\sin x} dx.$$

5. Let $f_n \in L^2((1,\infty), m)$, where m is the Lebesgue measure. Suppose that $\{f_n\}$ is a convergent sequence in $L^2((1,\infty), m)$. Prove that there is a measurable function f defined on $(1,\infty)$ such that f_n converges to f in measure.

6. Let $x = \{x_n\} \in l^1$ and

$$S = \{y = \{y_n\} \in l^1 : |y_n| \le |x_n|\}.$$

Prove that S is a compact subset of l^1.

7. Suppose that f is defined in $[a,b]$ and f' exists on $[a,b]$ except at finitely many points. Suppose that $|f'(x)| \le 1$ for $x \in [a,b]$ for which f' exists. Prove that f has bounded variation on $[a,b]$.

8. Prove that there exists a complex measure μ on $[0,1]$ such that

$$\int_{[0,1]} x^k d\mu = e^{k^2} \text{ for } k = 1, 2, \ldots 2003.$$

9. Let H be a Hilbert space with infinite dimension. Show that its linear dimension can not be countable.