Analysis Qualifying exam, Fall 2002.

Instructions. Partial credit will be given when appropriate. The decision on this examination will be based not only on the total point score, but also on whether the answers given are the result of careful thought and understanding.

Answer the questions as completely as possible. **Justify your work !!** If you use a major theorem, then cite it by name (e.g. by Fatou's Lemma), and check that its hypotheses are satisfied.

1. Let (X,M) be a measurable space. Let $\{\lambda_k\}, k \geq 1$, be a sequence of (positive) measures defined on M. For $E \in M$, define $\lambda(E) = \sum_{k=1}^{\infty} \lambda_k(E)$.

(a) Verify that λ is a measure on M.

(b) If f is a nonnegative measurable function on X, then show that

$$\int_X f \, d\lambda = \sum_{k=1}^{\infty} \int_X f \, d\lambda_k .$$

2. Let $\{c_{n,k}\}, n,k \geq 1$, be a set of complex numbers with the properties:

(i) $\lim_{n\to\infty} c_{n,k} = b_k$ for each $k \geq 1$;

(ii) $|c_{n,k}| \leq \frac{1}{k}$ for $n,k \geq 1$.

Show that $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k} c_{n,k} = \sum_{k=1}^{\infty} \frac{1}{k} b_k$.

3. Assume $\{f_n\} \subseteq L^1(\mathbf{R}), \{g_n\} \subseteq L^p(\mathbf{R})$ for some $p \geq 1$, with $\| f_n - f \|_1 \to 0$ and $\| g_n - g \|_p \to 0$ as $n \to \infty$. Also, assume that there exists $M > 0$ such that $| g_n | \leq M$ $a.e.$ for all $n \geq 1$.

(a) Prove that $fg \in L^1(\mathbf{R})$.

(b) Prove that there exists a subsequence $\{f_{n_k} g_{n_k}\}$ such that $\| f_{n_k} g_{n_k} - fg \|_1 \to 0$ as $k \to \infty$.

4. Let P be the collection of all subsets of $\{1,2,3,4,\ldots\}$. Let μ be counting measure, and let ν be the measure defined on P by $\nu(E) =$ the number of odd numbers in E (0 if there are none, ∞ if there is infinitely many). Characterize all finite (positive) measures λ on P with the properties:

$$\lambda << \mu \text{ and } \lambda \perp \nu .$$

5. Let $K(x,y)$ be a measurable function on $(0,\infty)\times(0,\infty)$ such that

$$B = ess\sup_{x\geq 0}\int_0^\infty |K(x,y)|^2 dy < \infty.$$

For $f \in L^2(0,\infty)$, define $T_K(f)$ by $T_K(f)(x) = \int_0^\infty K(x,y)f(y)\,dy$ for all x for which the integral exists.

(a) Show that T_K is a bounded linear operator from $L^2(0,\infty)$ into $L^\infty(0,\infty)$.

(b) Assume that S is a bounded linear operator from $L^1(0,\infty)$ into $L^2(0,\infty)$ such that $(S(g),f) = (g, T_K(f))$ for all $g \in L^1(0,\infty)$, all $f \in L^2(0,\infty)$. Here $(h,k) = \int_0^\infty h(x)k(x)\,dx$ if the integral exists. For $g \in L^1(0,\infty)$, find a formula for $S(g)$.

6. Let $\{f_1, f_2, \ldots, f_n\}$ be a linearly independent subset of $L^1[0,1]$. Show there exists $\{g_1, g_2, \ldots, g_n\} \subseteq L^\infty[0,1]$ such that $\int_0^1 f_k(x)g_j(x)\,dx = \left\{\begin{matrix} 1 \text{ if } j=k \\ 0 \text{ if } j\neq k\end{matrix}\right\}$.

7. Let $T : L^1[0,1] \to L^1[0,1]$ be a bounded linear map. Suppose that for all $f \in L^1[0,1]$, $T(f) \in C[0,1]$. Define $\widetilde{T} : L^1[0,1] \to C[0,1]$ (equipped with the usual uniform norm) by $\widetilde{T}(f) = T(f)$ for all $f \in L^1[0,1]$. Show that $\widetilde{T}$ is a bounded linear map from $L^1[0,1]$ into $C[0,1]$.

8. Assume that f and g are entire functions with $f(0) = f'(0) = 1$, $g(0) = g'(0) = 0$, $g''(0) = g'''(0) = 1$. Choose $r > 0$ such that g has no zeroes in $\{ z : 0 < |z| \leq r\}$. Let $\gamma(t) = re^{it}$ for $0 \leq t \leq 2\pi$. Compute $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{g(z)}\,dz$. [Do not do this problem for only two particular such functions f and g].

9. Let $\Omega = \{ z : \mathrm{Re}(z) > 0 \}$. For $g \in L^\infty(0,\infty)$, and $z \in \Omega$, set $G(z) = \int_0^\infty g(t)e^{-zt}dt$. Prove that $G(z)$ is holomorphic on Ω.

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1(a). It is clear that $\lambda(\phi) = 0$. Now let $\{E_j\}$, $j \geq 1$, be a disjoint sequence in M, and set $E = \cup_{j=1}^{\infty} E_j$. Then $\lambda(E) = \sum_{k=1}^{\infty} \lambda_k(E) = \sum_{k=1}^{\infty} \lambda_k(\cup_{j=1}^{\infty} E_j) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k(E_j) =$

$$\text{(Fubini)} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k(E_j) = \sum_{j=1}^{\infty} \lambda(E_j).$$

(b). Label the equality, $\int_X f \, d\lambda = \sum_{k=1}^{\infty} \int_X f \, d\lambda_k$, by (*). If $f = \chi_E$, then (*) holds by the definition of λ. Also, if f is a nonnegative measurable simple function, then (*) holds for f by the linearity of the integral. Now assume that f is a nonnegative measurable function. Choose a sequence of measurable simple functions, $\{s_n\}$, with $0 \leq s_n \leq f$ for all n, and $s_n \uparrow f$ (pointwise). Then by LMCT $\int_X f \, d\lambda = \lim_{n\to\infty} \int_X s_n \, d\lambda = \sum_{k=1}^{\infty} \int_X s_n \, d\lambda_k =$ (by LMCT) $\sum_{k=1}^{\infty} \int_X f \, d\lambda_k$. This proves (*) in the general case.

2. Set $f_n(k) = \frac{1}{k}c_{n,k}$, and $f(k) = \frac{1}{k}b_k$, $n,k \geq 1$. Then $f_n \to f$ pointwise as $n \to \infty$. Also,
$| f_n(k)| = \frac{1}{k}|c_{n,k}| \leq \frac{1}{k^2}$ for $n,k \geq 1$. Let μ be counting measure. The sequence $\{\frac{1}{k^2}\} \in L^1(\mu)$, so by LDCT, $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k}c_{n,k} = \lim\limits_{n\to\infty} \int_N f_n \, d\mu = \int_N f \, d\mu = \sum_{k=1}^{\infty} \frac{1}{k}b_k$.

3 (a). Since $\| g_n - g \|_2 \to 0$, there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$ a.e. Also,
$| g_{n_k} | \leq M$ a.e. for all $k \geq 1$. Therefore, $| g | \leq M$ a.e. Therefore, $fg \in L^1(\mathbf{R})$.
(b). Let $\{g_{n_k}\}$ be the same subsequence as in part (a). Then $| f_{n_k}g_{n_k} - fg | \leq |f|| g_{n_k} - g$ $|+$
$| g_{n_k} ||f_{n_k} - f| \leq |f|| g_{n_k} - g | + M|f_{n_k} - f|$. Now $|f|| g_{n_k} - g | \leq 2M|f| \in L^1(\mathbf{R})$, and $|f|| g_{n_k} - g | \to 0$ a.e. Therefore, by LDCT $\int_R |f|| g_{n_k} - g | \, dx \to 0$. Thus, $\| f_{n_k}g_{n_k} - fg$ $\|_1 \leq$
$\int_R |f|| g_{n_k} - g | \, dx + M\| f_{n_k} - f \|_1 \to 0$ as $k \to \infty$.

4. First, since $\lambda << \mu$ and λ is finite, by the Radon-Nikodym Thm. there exists a sequence $\{\delta_n\}$ with $\delta_n \geq 0$ for all n such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\lambda(E) = \sum_{n\in E} \delta_n$. Then since $\lambda \perp \nu$, for all odd numbers n, $\lambda(\{n\}) = 0$. Therefore, for all E, $\lambda(E) = \sum_{n\in E,\ n \text{ even}} \delta_n$.

5 (a). For $f \in L^2(0,\infty)$, $|T_K(f)(x)| \leq \int_0^{\infty} |K(x,y)||f(y)| \, dy \leq$ (by Cauchy-Schwarz) $(\int_0^{\infty} |K(x,y)|^2 \, dy)^{\frac{1}{2}} \| f \|_2$. It follows that $\| T_K(f)(x) \|_{\infty} \leq B^2 \| f \|_2$, so T_K is a bounded linear operator from $L^2(0,\infty)$ into $L^{\infty}(0,\infty)$.
(b). Let $g \in L^1(0,\infty)$, and $f \in L^2(0,\infty)$. Then $\int_0^{\infty} S(g)(x)f(x) \, dx = (S(g),f) = (g, T_K(f)) = \int_0^{\infty} g(y)(\int_0^{\infty} K(y,x)f(x) \, dx) \, dy =$ (Fubini) $\int_0^{\infty}(\int_0^{\infty} K(y,x)g(y) \, dy)f(x) \, dx$. Here Fubini is justified, since $\int_0^{\infty} |g(y)|(\int_0^{\infty} |K(y,x)||f(x)| \, dx) \, dy \leq \|g\|_1 \| \int_0^{\infty} g(y)(\int_0^{\infty} K(y,x)f(x) \, dx) \, dy \|_{\infty} \leq$
$\|g\|_1 B^2 \|f\|_2 < \infty$. Finally, since for every set E with $m(E) \leq \infty$, $\int_E S(g)(x) \, dx = \int_E(\int_0^{\infty} K(y,x)g(y) \, dy) \, dx$, we have $S(g)(x) = \int_0^{\infty} K(y,x)g(y) \, dy$ for all $g \in L^1(0,\infty)$.

6. For $1 \leq j \leq n$, let $M_j = span\{ f_k : 1 \leq k \leq n, \ j \neq k \}$. Since M_j is finite dimensional, it is a closed subspace of $L^1[0,1]$ and $f_j \notin M_j$. By a Corollary of Hahn-Banach, there exists a bounded linear functional α_j such that $\alpha_j(f_j) = 1$ and $\alpha_j(M_j) = \{0\}$. Now $(L^1)^* = L^{\infty}$, so there exist $g_j \in$

$L^\infty[0,1]$ such that $\alpha_j(f) = \int_0^1 f(x)g_j(x)\,dx$ for all $f \in L^1[0,1]$. The properties of the functionals α_j show that $\int_0^1 f_k(x)g_j(x)\,dx = \left\{ \begin{smallmatrix} 1 \text{ if } j=k \\ 0 \text{ if } j\neq k \end{smallmatrix} \right\}$.

7. Let $\widetilde{T} : L^1[0,1] \to C[0,1]$ be defined by $\widetilde{T}(f) = T(f)$. We show that $\widetilde{T}$ is a closed linear operator, and so is continuous by the CGT. Assume that $\{f_n\} \subseteq L^1[0,1]$, $\| f_n - f \|_1 \to 0$ and
$\|\widetilde{T}(f_n) - g \|_u \to 0$. Then $\| T(f_n) - T(f)\|_1 \to 0$ since T is bounded. There exists a subsequence $\{f_{n_k}\}$ such that $T(f_{n_k}) \to T(f)$ a.e. Since $\|\widetilde{T}(f_n) - g \|_u \to 0$, it follows that $g = T(f)$ a.e. Since g and $T(f)$ are continuous functions on $[0,1]$, $g = T(f)$.

8. $f(z) = 1 + z + \sum_{k=2}^\infty b_k z^k$, and $g(z) = \sum_{k=2}^\infty d_k z^k = z^2[\frac{1}{2} + \frac{1}{6}z + \sum_{k=4}^\infty d_k z^{k-2}]$ where the series converge for all z. Let $h(z) = \frac{1}{2} + \frac{1}{6}z + \sum_{k=4}^\infty d_k z^{k-2}$, so $g(z) = z^2 h(z)$. Let $D = D(0;r)$, and fix an open disk B such that $h(z) \neq 0$ on B and $\overline{D} \subseteq B$. Then $\frac{f(z)}{h(z)} = \sum_{k=0}^\infty c_k z^k$ for all $z \in B$. Now $\frac{f(z)}{g(z)} = \frac{c_0}{z^2} + \frac{c_1}{z} + \sum_{k=2}^\infty c_k z^{k-2}$, $z \in B$. Since $\mathrm{Res}(\frac{f}{g};0) = c_1$, we have by the Residue Thm. that $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{g(z)}\,dz = c_1$. Now $c_1 = (\frac{f}{h})'(0) = (f'(0)h(0) - f(0)h'(0))h(0)^{-2} = \frac{4}{3}$.

9. We prove that $G(z)$ is holomorphic on Ω by using Morera's Thm. First, assume that $\{z_n\} \subseteq \Omega$, $z_0 \in \Omega$, and $z_n \to z_0$. There exists δ with $0 < \delta \leq \mathrm{Re}(z_n)$ for $n \geq 1$. Then $g(t)e^{-z_n t} \to g(t)e^{-z_0 t}$ pointwise, and $|g(t)e^{-z_n t}| \leq |g(t)|e^{-\delta t} \in L^1(0,\infty)$. By LDCT, $G(z_n) = \int_0^\infty g(t)e^{-z_n t}dt \to \int_0^\infty g(t)e^{-z_0 t}dt = G(z)$.

Now let Δ be a triangle in Ω. Note that $\int_{\partial\Delta} e^{-zt}dz = 0$ by Cauchy's Thm. Then $\int_{\partial\Delta} G(z)dz = \int_{\partial\Delta}(\int_0^\infty g(t)e^{-zt}dt)dz =$ (Fubini) $\int_0^\infty g(t)(\int_{\partial\Delta} e^{-zt}dz)dt = 0$. To check that Fubini is justified, we use one side of a triangle in Ω. For $a,b \in \Omega$, let $\gamma(s) : [0,1] \to \Omega$ be $\gamma(s) = (1-s)a + sb$. Then $\int_\gamma(\int_0^\infty g(t)e^{-zt}dt)dz = \int_0^1(\int_0^\infty g(t)e^{-\gamma(s)t}dt)(b-a)ds$. It is straightforward to see that $g(t)e^{-\gamma(s)t} \in L^1([0,1] \times (0,\infty))$.