2014

Winter ~~2013~~ Algebra Qualifying Exam

THREE HOURS
The points will be split 30-78-72 between the three parts so divide your effort accordingly!

Part I

Give succinct answers to ALL of the following.
1. Let X be a set. Write down precisely the universal property that defines a *free group on set* X. Then EXPLAIN briefly why any two such are isomorphic.
2. State the Jacobson density theorem.
3. Give three different definitions/characterizations of a projective module.

Part II

Answer ALL of the following TRUE/FALSE questions, justifying your answer carefully to get any points.
1. There are exactly two isomorphism classes of finite groups of order 39.
2. $\mathbb{Z}/(25) \otimes_{\mathbb{Z}} \mathbb{Z}/(15) \cong \mathbb{Z}/(3)$.
3. The algebra A of upper triangular 3×3 matrices has just three indecomposable left modules (up to isomorphism), namely, the left ideals Ae_1, Ae_2 and Ae_3, where e_i is the diagonal matrix unit $e_{i,i}$.
4. For an integral domain R, every submodule of a finitely generated projective R-module is projective.
5. Any decreasing chain of algebraic subsets $X_1 \supseteq X_2 \supseteq \cdots$ in some affine algebraic variety X eventually stabilizes.
6. Working over an algebraically closed ground field K, let $\varphi : X \to Y$ be a morphism of affine varieties. If the comorphism $\varphi^* : K[Y] \to K[X]$ is injective then φ is surjective.

Part III

Answer any THREE of the following FOUR problems.
1. Find a set of representatives for the conjugacy classes in the group $GL_3(\mathbb{F}_2)$ and compute their sizes. Then show that this group is simple.
2. Recall for a ring R that a right R-module M is *flat* if the functor

$$M \otimes_R - : R\text{-}\boldsymbol{Mod} \to \boldsymbol{Ab}$$

is exact. Prove carefully that all projective right R-modules are flat. Then give one example of a module which is not flat.
3. Let A be a finite dimensional commutative algebra over an algebraically closed field K. Taking care to explain what theorems you are applying as you go, prove that there exist finite dimensional commutative *local* algebras $A_1, \ldots, A_n$ such that $A \cong A_1 \times \cdots \times A_n$. (Recall a commutative algebra is local if it has a unique maximal ideal.)

4. Let $f : V \to W$ be an R-module homomorphism for some commutative ring R. Show that f is injective if and only if the localized map $f_M : V_M \to W_M$ is injective for all maximal ideals M of R. Is the analogous statement with "injective" replaced with "surjective" true?