Your Name:

ALGEBRA QUALIFYING EXAMINATION, WINTER 2011

R always means a ring with 1.

Part I. (6 points each):

1. State the Jordan-Hölder theorem for finite groups. Give definitions.
2. Give at least three equivalent definitions of the Jacobson radical $J(R)$ of a ring R (changing 'left' to 'right' or vice versa does not count as a new definition).
3. Let R, S and T be rings; let V be an (R, S)-bimodule and W an (S, T)-bimodule. Give a definition of the tensor product $V \otimes_S W$, explain how it is an (R, T)-bimodule, and state its universal property.

Part II. True or false? If true provide a brief explanation, if false provide a counterexample (6 points each):

1. The group G below is trivial.
$$G = < a, b, c | bab = a^2, bcb = c^2, b^2c = cb^2, a^3 = c^3 > .$$
2. If V is a left R-module and $X \subset V$ is a subset then $Ann_R(X) = \{r \in R | rX = 0\}$ is a two-sided ideal of R.
3. The group $(\mathbb{R}, +)$ has no proper subgroups of finite index.
4. There exists a ring R and a non-zero cyclic left R-module V such that $V \oplus V$ is also cyclic.
5. If $R = F[x]/(p(x))$ where F is a field and $p(x)$ is a polynomial of degree two then $R/J(R)$ is a division ring.
6. Let V be a finite dimensional linear space over $\mathbb{C}$ and $\Lambda(V)$ the exterior algebra of V. Then for any element x in the maximal ideal $M = \oplus_{i>0}\Lambda^i(V)$ we have $x^2 (= x \wedge x) = 0$.
7. Let R be commutative and V an R-module. Then $V = 0$ if and only if the localization $V_M = 0$ for every maximal ideal M of R.

Part III. Give detailed solutions of the problems (10 points each):

1. Classify the groups of order 175.
2. Let P be a projective R-module and $S = \{p_1, p_2, \ldots\}$ a generating system of P. Then there exists a subset $\{p_1^*, p_2^*, \ldots\}$ of $P^* = \mathrm{Hom}_R(P, R)$ such that for every $x \in P$ we have $x = \sum_i p_i^*(x)p_i$ where $p_i^*(x) = 0$ for all but finitely many i.

3. Classify the semisimple $\mathbb{R}$-algebras of dimension 7. You may assume without proof the fact that the only division algebras over $\mathbb{R}$ up to isomorphism are $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ (the quaternion algebra).
4. Let $I = (x^2y^2, x^2z^3)$ be the ideal of $\mathbb{C}[x, y, z]$. Find its radical, the minimal associated primes, a primary decomposition of I, and the embedded primes.