ALGEBRA QUALIFYING EXAMINATION, WINTER 201}

Your Name:

Part I. Carefully state each of the following (6 points each):

1. Classification of finitely generated modules over PIDs.
2. Two equivalent definitions of an injective module.
3. Explain what are unit and counit of adjunction and what do they have to do with adjoint functors.

Part II. True or false? If true provide a brief explanation, if false provide a counterexample (6 points each):

1. If F is a finite field and $a, b \in F$ are not squares then ab is a square.
2. Let R be a ring and I, J be two left ideals in R. If the modules R/I and R/J are isomorphic then $I = J$.
3. Let G be a finite group such that every irreducible $\mathbb{C}G$-module is one-dimensional. Then G is abelian.
4. If R is a commutative noetherian local ring with maximal ideal M which satisfies $M^2 = M$ then R is a field.
5. Let F be an algebraically closed field of characteristic $p > 0$, and $\mathrm{Fr} : \mathbb{A}^1 \to \mathbb{A}^1, a \mapsto a^p$ be the Frobenius homomorphism. True or false:
   (a) Fr is a homeomorphism in the Zariski topology.
   (b) Fr is an isomorphism of algebraic sets.

Part III. Attempt any four of the following five problems (13 points each; you can only get credit for four):

1. Suppose that a finite group G has exactly three Sylow p-subgroups. Show that every permutation of these Sylow subgroups can be obtained by conjugation by some suitable element of G.
2. Let V be an R-module. A family $(V_i)_{i\in I}$ of submodules of V is called a *directed system of submodules* if for any $i, j \in I$ there exists $k \in I$ such that $V_i \subseteq V_k$ and $V_j \subseteq V_k$. Prove that V is finitely generated if and only if the union $\cup_{i\in I} V_i$ of any directed set of proper submodules is proper.
3. Let K/F be a field extension and $A, B \in M_n(F)$. If A and B are similar as matrices over K then they are similar as matrices over F.

4. Let H be a normal subgroup of a finite group G, $g \in G$, $\bar{G} := G/H$ and $\bar{g} := gH \in \bar{G}$. Then $|C_{\bar{G}}(\bar{g})| \leq |C_G(g)|$.
5. Let $m \geq 3$, φ be the Euler function, and $\varepsilon \in \mathbb{C}$ be a primitive mth root of 1. Prove that $[\mathbb{Q}(\varepsilon + \varepsilon^{-1}) : \mathbb{Q}] = \varphi(m)/2$.