Algebra Qualifying Exam - Winter 2010

General Directions - You should assume that all rings have an identity and that all modules are unitary.

Part I: Definitions and Theorems. (6 points each.)

1. State any 3 equivalent definitions of *injective module.*

2. State Noether's Normalization Theorem.

Part II: Determine if each statement is TRUE or FALSE. Give a brief justification. (8 points each.)

1. $\mathbb{C}[x,y]/(x^2-3y)$ and $\mathbb{C}[x,y]/(y^2-3x)$ are isomorphic $\mathbb{C}[x,y]$-modules.

2. Let I be an ideal in the ring R such that $IL \neq L$ for every simple left R-module L. Then $IM \neq M$ for every nonzero finitely generated left R-module M.

3. An Artinian ring has a finite number of simple modules (up to isomorphism).

4. $(x^3, 4x, 8)$ is primary in $\mathbb{Z}[x]$.

5. A non-Abelian group of order 55 has exactly 11 one-dimensional complex representations, up to isomorphism.

Part III: Give complete solutions to FOUR of the following FIVE problems. (12 points each)

1. Let R be an integral domain considered as a subring of its quotient field Q. Prove: $\cap_{M\in Max(R)} R_M = R$.

2. Let M be an R-module and assume that every short exact sequence of R-modules $0 \to A \to M \to B \to 0$ is split exact. Prove directly that M is the sum of its simple submodules.

3. Explicitly construct three pairwise non-isomorphic groups of order 147 which do not have any elements of order 49.

4. Let M be a Noetherian module for the ring R and $f : M \to M$ a surjective homomorphism. Prove that f is an isomorphism.

5. Let σ and ϕ be the elements of $Aut_{\mathbb{Q}}(\mathbb{Q}(x))$ defined by $\sigma(x) = \frac{1}{x}$ and $\phi(x) = \frac{1}{1-x}$. Let G be the subgroup of $Aut_{\mathbb{Q}}(\mathbb{Q}(x))$ generated by σ and ϕ and let K be the fixed field of G. Calculate $[\mathbb{Q}(x) : K]$ and find all of the proper subfields between K and $\mathbb{Q}(x)$, explicitly. State clearly which intermediate fields are Galois over K and which are not.

Algebra Qualifying Exam - Winter 2010 - Solutions

Part II: Determine if each statement is TRUE or FALSE. Give a brief justification. (8 points each.)

1. $\mathbb{C}[x,y]/(x^2-3y)$ and $\mathbb{C}[x,y]/(y^2-3x)$ are isomorphic $\mathbb{C}[x,y]$-modules.

False: These modules have different annihilators.

2. Let I be an ideal in the ring R such that $IL \neq L$ for every simple left R-module L. Then $IM \neq M$ for every nonzero finitely generated left R-module M.

True: Since I must be contained in $J(R)$, this is a slight reformulation of the (noncommutative) NAK lemma.

3. An Artinian ring has a finite number of simple modules (up to isomorphism).

True: R and $R/J(R)$ have the same simple modules and the statement is true for $R/J(R)$ by Artin-Wedderburn.

4. $(x^3, 4x, 8)$ is primary in $\mathbb{Z}[x]$.

True: The radical is $(x, 2)$, a maximal ideal.

5. A non-Abelian group of order 55 has exactly 11 one-dimensional complex representations, up to isomorphism.

False: $|G/G'| = 5$ so it has 5 one-dimensional representations.

Part III: Give complete solutions to FOUR of the following FIVE problems. (12 points each)

1. Let R be an integral domain considered as a subring of its quotient field Q. Prove: $\cap_{M \in Max(R)} R_M = R$.

Since R is an integral domain, $R \subset R_M \subset Q$ for all $M \in Max(R)$. Let q be an element of $\cap_{M \in Max(R)} R_M$ and set $I_q = \{r \in R \,|\, rq \in R\}$. Then I_q is an ideal of R and $q \in R$ if an only if $I_q = R$ (i.e. $1 \in I_q$). If I_q is proper in R then it is contained in some maximal ideal M. But this means that for every representation of q as a/b, $a, b \in R$, b must be in M, contradicting the assumption $q \in R_M$. Hence $I_q = R$ and $q \in R$, as required.

2. Let M be an R-module and assume that every short exact sequence of R-modules $0 \to A \to M \to B \to 0$ is split exact. Prove directly that M is the sum of its simple submodules.

An application of Zorn's lemma assures that M has a maximal proper submodule. Splitting the resulting short exact sequence shows that M has a simple submodule. Hence the set of all sums of simple submodules of M is non-empty. A second application of Zorn's lemma yields a maximal element of the set, say M'. If M' is not M, the splting the resulting short exact sequence yields a non-zero submodule N with $M = M' \oplus N$. We see at once that N satisfies the same hypothesis as M and hence also has a simple submodule, which is not contained in M'. The contradicts the maximality of M' and proves that $M' = M$.

3. Explicitly construct three pairwise non-isomorphic groups of order 147 which do not have any elements of order 49.

$147 = 3 \cdot 7^2$. The 7-Sylow subgroup is therefore normal and Abelian and must be $H := \mathbb{Z}_7 \times \mathbb{Z}_7$. Any 3-Sylow subgroup must be $\mathbb{Z}_3$ and the group is a semidirect product $\mathbb{Z}_3 \times_\phi H$ for some $\phi : \mathbb{Z}_3 \to Aut(H) = GL_2(\mathbb{Z}_7)$. Over $\mathbb{Z}_7$, the polynomial $x^3 - 1$ splits as $(x-1)(x-2)(x-4)$. Thus any generator of $\mathbb{Z}_3$ in $AUt(H)$ is conjugate to one of the diagonal matrices $diag(1,1)$, $diag(1,2)$, $diag(1,4)$, $diag(2,2)$, $diag(2,4)$, $diag(4,4)$.

4. Let M be a Noetherian module for the ring R and $f : M \to M$ a surjective homomorphism. Prove that f is an isomorphism.

Since M is Noetherian, for some n, $ker(f^n) = ker(f^{n+k})$ for all $k \geq 0$. Let $x \in ker(f)$. Since f is surjective, so is f^n and we may choose y for which $x = f^n(y)$. Then $0 = f(x) = f^{n+1}(y)$ implies $0 = f^n(y) = x$. Hence $ker(f) = 0$.

5. Let σ and ϕ be the elements of $Aut_{\mathbb{Q}}(\mathbb{Q}(x))$ defined by $\sigma(x) = \frac{1}{x}$ and $\phi(x) = \frac{1}{1-x}$. Let G be the subgroup of $Aut_{\mathbb{Q}}(\mathbb{Q}(x))$ generated by σ and ϕ and let K be the fixed field of G. Calculate $[\mathbb{Q}(x) : K]$ and find all of the proper subfields between K and $\mathbb{Q}(x)$, explicitly. State clearly which intermediate fields are Galois over K and which are not.

By direct calculation we see that $\sigma^2 = \phi^3 = I$ and $\sigma\phi = \phi^2\sigma$, so the group G is D_3. Thus $\mathbb{Q}(x)$ is algebraic and Galois over K and $[G : K] = 6$. Note: $\phi^2(x) = \frac{x-1}{x}$, $\sigma\phi(x) = \frac{x}{x-1}$ and $\sigma\phi^2(x) = 1 - x$. G has four nontrivial proper subgroups, $\langle\sigma\rangle$, $\langle\sigma\phi\rangle$, $\langle\sigma\phi^2\rangle$, and $\langle\phi\rangle$. These correspond, respectively, to the fixed fields $\mathbb{Q}(u_1)$, $\mathbb{Q}(u_2)$, $\mathbb{Q}(u_3)$ and $\mathbb{Q}(v)$ where $u_1 = x + \sigma(x) = (x^2+1)/x$, $u_2 = x \cdot \sigma\phi(x) = x^2/(x-1)$, $u_3 = x \cdot \sigma\tau^2(x) = x - x^2$ and $v = x + \tau(x) + \tau^2(x) = (x^3 - 2x^2 + 3x - 1)/(x - x^2)$. These are all of the proper intermediate fields and only the last one is Galois over K since only the last subgroup is Normal in G.

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