Qualifying Exam in Algebra
Winter 2009

Part I. Definitions and theorems.
1. (6 points). Give a definition of a finite Galois extension.
2. (6 points). Formulate the theorem describing the structure of semisimple finite-dimensional algebras.

Part II. True or false. Give brief justification.
1. (8 points). The cyclotomic field $\mathbb{Q}(\sqrt[5]{1})$ contains $\sqrt{5}$.
2. (8 points). A matrix A in $\mathrm{Mat}_2(\mathbb{C})$ such that $\mathrm{Tr}(A^k)$ is an integer for all $k \geq 0$ has integer coefficients.
3. (8 points). For every nonzero ideal $I \subset \mathbb{Z}[\sqrt{2}]$ the set $\mathbb{Z}[\sqrt{2}]/I$ is finite.
4. (8 points). Let $f(t)$ be a separable irreducible polynomial of degree n over some field. Then the Galois group of f contains a cycle of length n.
5. (8 points). Let I and J be radical ideals of $\mathbb{C}[x, y]$. Then $I + J$ is also radical.

Part III. Longer problems. Solve four of the following five problems.
1. (12 points). Let R be a ring. Suppose a left R-module M is a sum of finitely many submodules M_i. Prove that M is Noetherian if and only if each of M_i is.
2. (12 points). Let F be a degree 6 field extension of $\mathbb{Q}$. What is the maximal possible order of a finite subgroup of F^*?
3. (12 points). Let G be a nonabelian group of order 21. Find the dimensions of irreducible complex representations of G.
4. (12 points). Let A and B be two non-isomorphic simple left R-modules. Prove that the module $A \oplus B$ is cyclic.
5. (12 points). Let $\mathbb{H}$ denote the quaternion algebra. Construct explicitly:
(i) an isomorphism of $\mathbb{R}$-algebras $\mathbb{H} \simeq \mathbb{H}^{opp}$;
(ii) an isomorphism of $\mathbb{C}$-algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Mat}_2(\mathbb{C})$.

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Solutions

Part I. Definitions and theorems.
1. (6 points). Give a definition of a finite Galois extension.
2. (6 points). Formulate the theorem describing the structure of semisimple finite-dimensional algebras.

Part II. True or false. Give brief justification.
1. (8 points). The cyclotomic field $\mathbb{Q}(\sqrt[5]{1})$ contains $\sqrt{5}$.

Solution: True. Let ζ be a primitive 5-th root of unity (it is contained in this field). Then the element $a = \zeta + \zeta^{-1}$ is quadratic over $\mathbb{Q}$ with the conjugate $a' = \zeta^2 + \zeta^{-2}$. We have $a + a' = -1$ and $aa' = \zeta^3 + \zeta^{-1} + \zeta + \zeta^{-3} = -1$, so $a^2 + a - 1 = 0$. The discriminant of this quadratic equation is 5, so $\sqrt{5}$ can be expressed in terms of a.

2. (8 points). A matrix A in $\mathrm{Mat}_2(\mathbb{C})$ such that $\mathrm{Tr}(A^k)$ is an integer for all $k \geq 0$ has integer coefficients.

Solution: False. Take the matrix $A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$. Then $\mathrm{Tr}(A^k) = 0$ for all $k \geq 0$.

3. (8 points). For every nonzero ideal $I \subset \mathbb{Z}[\sqrt{2}]$ the set $\mathbb{Z}[\sqrt{2}]/I$ is finite.

Solution: True. It is enough to check that the intersection $I \cap \mathbb{Z}$ is nonzero (since $\mathbb{Z}[\sqrt{2}]/I$ is a finitely generated module over $\mathbb{Z}/(I \cap \mathbb{Z})$). Assume this intersection is zero. Then I corresponds to an ideal in $\mathbb{Q}[\sqrt{2}]$ (by the theorem on ideals in a localized ring). But $\mathbb{Q}[\sqrt{2}]$ is a field, so $I = 0$ - contradiction.

4. (8 points). Let $f(t)$ be a separable irreducible polynomial of degree n over some field. Then the Galois group of f contains a cycle of length n.

Solution: False. There exists a quartic polynomial with the Galois group being the Klein 4-subgroup in S_4. This subgroup has no elements of order 4.

5. (8 points). Let I and J be radical ideals of $\mathbb{C}[x, y]$. Then $I + J$ is also radical.

Solution: False. Take $I = (x)$, $J = (x - y^2)$ (both prime). Then $I + J = (x, y^2)$ is not radical.

Part III. Longer problems. Solve four of the following five problems.
1. (12 points). Let R be a ring. Suppose a left R-module M is a sum of finitely many submodules M_i. Prove that M is Noetherian if and only if each of M_i is.

Solution: A submodule of a Noetherian module is Noetherian, so if M is Noetherian then so are M_i. The converse can be proved by induction in the number of M_i's. Indeed, assume $M = M_1 + \ldots + M_n$. It is enough to check that $M_1 + M_2 \subset M$ is Noetherian. But this follows from the exact sequence $0 \to M_1 \to M_1 + M_2 \to M_2/M_1 \cap M_2 \to 0$, since both M_1 and $M_2/M_1 \cap M_2$ are Noetherian.

2. (12 points). Let F be a degree 6 field extension of $\mathbb{Q}$. What is the maximal possible order of a finite subgroup of F^*?

Answer: n=18. Solution: such a subgroup would have to consist of roots of unity, so it would be generated by some primitive nth root of unity. Hence, the question is really to find the maximal n such that the degree of $[\mathbb{Q}(\sqrt[n]{1}) : \mathbb{Q}]$ divides 6. But this degree equals $\phi(n)$, so we have to find maximal n such that $\phi(n)|6$. Let $n = p_1^{a_1} \ldots p_k^{a_k}$, where $p_1, \ldots, p_k$ are primes. Then $\phi(n) = p_1 a_1 - 1 \ldots p_k^{a_k - 1}(p_1 - 1) \ldots (p_k - 1)$. Hence, the only primes that can divide n are 2, 3, and 7. If $7|n$ then $n = 7$. Otherwise, $n = 2^k 3^m$ with $k \leq 2$ and $m \leq 2$, so the answer is easy to find by looking at few remaining cases.

3. (12 points). Let G be a nonabelian group of order 21. Find the dimensions of irreducible complex representations of G.

Solution: By Sylow's theorems G has a normal subgroup of order 7. Hence, the commutant of G coincides with this subgroup, so G has exactly 3 one-dimensional characters. Since the dimensions of irreducible representations divide the order of G, the other irreducible representations should have odd dimension ≥ 3 and their squares sum up to $18 = 21 - 3$. Hence, G has two irreducible representations of dimension 3.

4. (12 points). Let A and B be two non-isomorphic simple left R-modules. Prove that the module $A \oplus B$ is cyclic.

Solution: Pick nonzero elements $a \in A$ and $b \in B$. Let $C \subset A \oplus B$ be the submodule generated by (a, b). Assume $C \cap A = 0$. Then the projection gives an isomorphism $C \to B$ (since B is simple). Therefore, $\text{Hom}_R(C, A) = 0$ by Schur's lemma, so $C = 0 \oplus B$, which is contradiction. Hence, $C \cap A = A$. Similarly, $C \cap B = B$, so $C = A \oplus B$.

5. (12 points). Let $\mathbb{H}$ denote the quaternion algebra. Construct explicitly:

(i) an isomorphism of $\mathbb{R}$-algebras $\mathbb{H} \simeq \mathbb{H}^{opp}$;

(ii) an isomorphism of $\mathbb{C}$-algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \text{Mat}_2(\mathbb{C})$.

Solution: (i) $a + bi + cj + dk \mapsto a - bi - cj - dk$. (ii) send $i \otimes 1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j \otimes 1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $k \otimes 1 \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.