Qualifying Exam in Algebra, Winter 200~~7~~8.

Part I. Definitions and Theorems.

1 (6 points). Give two equivalent definitions of projective module.

2 (6 points). State the Primitive Element Theorem from the theory of field extensions.

Part II. True or false. Give brief justification.

1 (8 points). The center of a nontrivial solvable group is nontrivial.

2 (8 points). Let $f(x)$ be an irreducible separable polynomial of degree n. Then $|Gal(f(x))|$ is divisible by n.

3 (8 points). Let G be a finite group such that the group $Aut(G)$ is cyclic. Then G is abelian.

4 (8 points). The ring $\mathbb{C}[x,y]$ is Jacobson semisimple.

5 (8 points). Let $R = \mathbb{F}_{25}[x]$. Then $K_0(R) = \mathbb{Z}$.

Part III. Longer problems.

You have to solve any 4 of the problems below.

1 (12 points). Let I and J be two ideals of $\mathbb{C}[x_1, \ldots, x_n]$. Assume that IJ is radical. Prove that $IJ = I \cap J$.

2 (12 points). Let A be a noncommutative finite dimensional algebra over $\mathbb{C}$ such that the length of $A-$module A is 2. What is $dim(A)$?

3 (12 points). Find the Galois group of the polynomial $x^4 + x^2 + 1$ over $\mathbb{Q}$.

4 (12 points). Let V be a linear space of dimension n and $F : V \to V$ be a linear operator with determinant D. What is determinant of $\bigwedge^2 F : \bigwedge^2 V \to \bigwedge^2 V$?

5 (12 points). Let G be a finite group with precisely 5 inequivalent irreducible representations of dimension 1,1,2,3 and d. Find d.

Qualifying Exam in Algebra, Winter $200\cancel{7}$ 8. Solutions

Part I. Definitions and Theorems.

1 (6 points). Give two equivalent definitions of projective module.

Answer: P is projective if

1) for any exact sequence $A \xrightarrow{\phi} B \to 0$ and a homomorphism $f : P \to B$ there exists a homomorphism $g : P \to A$ such that $f = \phi \circ g$.
2) The functor $\mathrm{Hom}(P, ?)$ is exact.
3) P is a direct summand of a free module.

2 (6 points). State the Primitive Element Theorem from the theory of field extensions.

Answer: A finite separable extension E/K can be generated by one element: $E = K(\alpha)$.

Part II. True or false. Give brief justification.

1 (8 points). The center of a nontrivial solvable group is nontrivial.

Solution. False. Group S_3 is solvable with trivial center.

2 (8 points). Let $f(x)$ be an irreducible separable polynomial of degree n. Then $|Gal(f(x))|$ is divisible by n.

Solution. True. The action of $Gal(f(x))$ on the roots of $f(x)$ is transitive, hence $|Gal(f(x))| = n \cdot |St|$ where $St \subset Gal(f(x))$ is a stabilizer of a root.

3 (8 points). Let G be a finite group such that the group $Aut(G)$ is cyclic. Then G is abelian.

Solution. True. Group $G/Z(G)$ is a subgroup of $Aut(G)$ (inner automorphisms). A subgroup of a cyclic group is cyclic and thus $G/Z(G)$ should be cyclic. Thus $G/Z(G)$ is trivial.

4 (8 points). The ring $\mathbb{C}[x, y]$ is Jacobson semisimple.

Solution. True. The Jacobson radical is an intersection of all maximal ideals. By Nullstellensatz the maximal ideals are of the form $\{f \in \mathbb{C}[x, y] | f(a, b) = 0\}$ for various $(a, b) \in \mathbb{C}^2$. Clearly, the intersection is zero.

5 (8 points). Let $R = \mathbb{F}_{25}[x]$. Then $K_0(R) = \mathbb{Z}$.

Solution. True. The ring R is PID, hence any finitely generated projective module is free; free modules over domain are (stably) isomorphic only if ranks coincide.

Part III. Longer problems.

You have to solve any 4 of the problems below.

1 (12 points). Let I and J be two ideals of $\mathbb{C}[x_1, \dots, x_n]$. Assume that IJ is radical. Prove that $IJ = I \cap J$.

Solution. We have $IJ \subset I \cap J \subset \sqrt{I \cap J} = \sqrt{IJ} = IJ$. The result follows.

2 (12 points). Let A be a noncommutative finite dimensional algebra over $\mathbb{C}$ such that the length of $A-$module A is 2. What is $dim(A)$?

Solution. First assume that $J(A) \neq 0$. Then $J(A)$ and $A/J(A)$ should be simple $A-$modules. Thus $A/J(A)$ is simple $A/J(A)-$module and hence $A/J(A) = \mathbb{C}$. Since $J(A)$ is simple over $A/J(A)$ it is one dimensional over $\mathbb{C}$. Thus A is two dimensional and hence commutative. Thus $J(A) = 0$ and A is semisimple. We have two possibilities: $A = \mathbb{C} \oplus \mathbb{C}$ and $A = Mat_2(\mathbb{C})$. First is again commutative. Thus $A = Mat_2(\mathbb{C})$.

Answer: $dim(A) = 4$.

3 (12 points). Find the Galois group of the polynomial $x^4 + x^2 + 1$ over $\mathbb{Q}$.

Solution. We have $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 - x + 1)(x^2 + x + 1)$. The roots of $x^2 - x + 1$ and $x^2 + x + 1$ differ only by sign, hence generate the same quadratic field. Hence $Gal(x^4 + x^2 + 1) = Gal(x^2 - x + 1) = \mathbb{Z}/2\mathbb{Z}$.

Answer: $Gal(x^4 + x^2 + 1) = \mathbb{Z}/2\mathbb{Z}$.

4 (12 points). Let V be a linear space of dimension n and $F : V \to V$ be a linear operator with determinant D. What is determinant of $\bigwedge^2 F : \bigwedge^2 V \to \bigwedge^2 V$?

Solution. We can assume that the base field is algebraically closed. Using Jordan normal from we can choose a basis $\{e_1, \dots, e_n\}$ such that F is upper triangular with eigenvalues $\lambda_1, \dots, \lambda_n$. Now in basis $e_1 \wedge e_2, \dots, e_1 \wedge e_n, e_2 \wedge e_3, \dots, e_{n-1} \wedge e_n$ the operator $\bigwedge^2 F$ is upper triangular with eigenvalues $\lambda_1\lambda_2, \dots, \lambda_{n-1}\lambda_n$. Hence determinant of $\bigwedge^2 F$ is $\prod_{i<j} \lambda_i\lambda_j = (\prod_i \lambda_i)^{n-1} = D^{n-1}$.

Answer: D^{n-1}.

5 (12 points). Let G be a finite group with precisely 5 inequivalent irreducible representations of dimension 1,1,2,3 and d. Find d.

Solution. We have $|G| = 1^2 + 1^2 + 2^2 + 3^2 + d^2 = 15 + d^2$. Since $|G|$ should be divisible by $2, 3$ and d we find that $d = 3$ or $d = 15$. Assume that $d = 15$ and $|G| = 240$. Then $|G : G'| = 2$ (since G has precisely 2 linear characters) and G' contains at least 4 conjugacy classes of G: elements of order 1,2,3,5. Thus $G - G'$ should be a single conjugacy class of size 120. Thus for $x \in G - G'$ we have $|C_G(x)| = 2$. This implies $x^2 = 1$ (since $1, x, x^2 \in C_G(x)$) and hence x lies in a Sylow 2-subgroup P. But then $|C_G(x)| > 2$: either x is not central in P and then $C_G(x)$ contains x and $Z(P)$, or x is central and $C_G(x)$ contains P. This is a contradiction. Hence $d = 3$ (this is really possible; $G = S_4$ is an example).

Answer: $d = 3$.

You have to solve any 4 of the problems below.

1 (12 points). Let I and J be two ideals of $\mathbb{C}[x_1, \ldots, x_n]$. Assume that IJ is radical. Prove that $IJ = I \cap J$.

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Answer: $dim(A) = 4$.

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Answer: $d = 3$.