ALGEBRA QUALIFYING EXAM   Winter 2007

I. Definitions and Theorems

1. (a) Define adjoint functors.
   (b) State the theorem about adjointness of the functors Hom and $\otimes$ between categories of left modules over two rings.
2. (a) Give three equivalent definitions of the Jacobson radical of a ring.
   (b) State the theorem about the Jacobson radical of Artinian rings.
3. State the orthogonality properties of characters of finite groups.

II. True or False questions. Give brief but to the point justification.

1. There are no simple groups of order 200.
2. Every algebraic extension of the finite field $\mathbf{F}_p$ (p is prime) is separable.
3. Let R be a PID and M be a cyclic R-module of order p^n, where $p \in R$ is prime. Then M is indecomposable.
4. If two matrices in $M_4(\mathbb{C})$ have equal characteristic and minimal polynomials then they are similar.
5. Every Artinian integral domain is a field.
6. If modules $M \in \mathsf{Mod}\text{-}R$ and $N \in R\text{-}\mathsf{Mod}$ are flat then $M \otimes_R N$ is a flat abelian group.

III. Longer problems. Do any *four* of the following problems.

1. Let G be a finite group and $H \subset G$ a subgroup of index p, where p is the smallest prime dividing $|G|$. Show that H is normal.
2. Let $\alpha \in \mathbb{R}$ be a root of $f \in \mathbb{Q}[x]$ with the splitting field K_f of degree $[K_f : \mathbb{Q}] = 2^n$ for some $n \in \mathbb{N}$. Prove that α is a constructible number (by the straightedge and compass).
3. Let V be a vector space of dimension n and $f \in \mathrm{End}_k(V)$. Let $\Lambda^2(f)$ be the endomorphisms of the space $\Lambda^2(V)$ induced by f.

   Prove that the trace of $\Lambda^2(f)$ is equal to the coefficient of the characteristic polynomial $\chi_f = \det(t \cdot Id - f)$ at t^{n-2}.
4. Prove that the group algebras $\mathbb{C}[G]$ and $\mathbb{C}[H]$ of finite groups G and H are Morita equivalent if and only if G and H have the same number of conjugacy classes.
5. Prove that if for $f, g \in \mathbb{C}[x, y]$ the system of equations $f(x, y) = 0,\ g(x, y) = 0$ has finitely many solutions in $\mathbb{C}^2$, then $\mathbb{C}[x, y]/(f, g)$ is a finite-dimensional algebra over $\mathbb{C}$.